# Hex

Everything You Always Wanted to Know About Hex
But Were Afraid to Ask

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# Contents

# Chapter 1

# Introduction

Hex is a two-player game played on a rhombic board with a hexagonal pattern of cells. The rules are simple: Black and White take turns occupying a cell and the one to form a chain connecting his two opposite sides, and thereby blocking the other from connecting his sides, wins.

Out of these simple rules arises a game that is surprisingly difficult to play well. It has a number of interesting mathematical features and is the subject of modern artificial intelligence research. Even its history is worth looking into as it was discovered first by the Danish scientist, artist and poet Piet Hein and rediscovered only a few years later by the American mathematician and Nobel prize winner John Nash. Belonging to the spheres of both games and mathematics it has enjoyed great attention from people like Martin Gardner and the late French mathematician Claude Berge.

This paper is a thesis for a Master of Science degree at the Department of Mathematics and Computer Science, University of Southern Denmark. The objective of the thesis is to provide a description of the board game Hex, its mathematical aspects and its history.

The thesis may not be Everything You Always Wanted to Know About Hex—But Were Afraid to Ask[1]—but it is the so far most comprehensive account of the history of Hex and brings to light new details about the circumstances of its invention. It is also the first time that all the aspects of the game are collected into one document.

The text is divided into six main chapters each concerning an independently interesting aspect of the game of Hex.

**The History** of Hex is interesting mainly because of the personages behind Hex, but also because it is available and yet until now not fully discovered. Some of Piet Hein's original manuscripts have been brought to light for the first time and parts of his columns in the Danish newspaper *Politiken* are reviewed.

[1]Everything You Always Wanted to Know About Sex—But Were Afraid to Ask is a film by Woody Allen.

**Game theory** in Hex includes a proof of existence of the first player's winning strategy. The necessary properties are easily ascertained but are not as easily proven. The present proof relies mostly on graph theory.

**The complexity** of Hex is known to be deep. We shall show Hex to belong to PSPACE by a series of transformations from a quantified boolean formula.

**Playing Hex** is simple; playing Hex well is much more complicated. Some advice is useful.

**Variants** of Hex offer new insights and experiences. The beauty and simplicity of Hex can be found in different games; by looking into other games we shall discover the properties of Hex.

**Recent research** revolves around creating the best artificial Hex player. Highly different approaches to this challenge are being taken and each brings new knowledge.

Appendix A contains some of my efforts to communicate the wonders of Hex to a wider audience than the readers of this document. I have written a website, a popular article and commenced a textbook on the subject.

Appendix B contains a comprehensive list of Hex in the Danish newspaper as well as the first translations of some of Piet Hein's interesting unpublished manuscripts.

In my work I use concepts and terms and even a few results from game theory, graph theory and general mathematics. It is not the purpose of this thesis to define these and so the reader is expected to have some knowledge of the areas. All notation, terminology, lemmas and theorems can be found in elementary textbooks on the respective subjects.

All that remains now is that I thank all the people who have been helpful during this year's work. I was fortunate that John F. Nash, Jr., David Gale, Martin Gardner, Harold W. Kuhn were able to help me establish the correct history at Princeton University. Aage Bohr provided details on *The Parenthesis*. Hugo Piet Hein keeps an impressive archive of his father's work and I am grateful for being allowed to look into it as well as for the help he provided. Other people have aided and some have merely been interested and I thank all for making the work pleasant.

I must thank also my advisor Bjarne Toft for his guidance and for never losing interest. I hope that he does not take offence that I beat him even as second player.

Thomas Maarup
Odense, May 27, 2005[2]

[2] The present version has been approved and had minor errors fixed.

# Chapter 2

# Resume på dansk

Dette speciale ved Institut for Matematik og Datalogi, Syddansk Universitet har titlen *Hex—Alt du gerne ville vide om Hex, men ikke turde spørge om*.[1] Det er en omfattende gennemgang af brætspillet Hex, dets historie og matematiske aspekter, herunder spilteori og grafteori.

Hex er et spil for to spillere på et rombisk bræt inddelt i $n \times n$ sekskanter. Spillerne ejer hver et par modstående sider som de forsøger at forbinde ved skiftevis at besætte et felt. Vinder er den som dette lykkes for.

Spillet er opfundet af Piet Hein i 1942 og offentliggjort i Politiken i en regelmæssig klumme i 1942–43. Seks år senere opfandt John Nash spillet, uafhængigt af Hein, på Princeton Universitet. Siden beskrev Martin Gardner Hex i sin klumme om matematiske spil i *Scientific American* hvorfra det blev kendt internationalt.

Særligt interessant er spillet fordi der altid er netop en vinder og da det er endeligt, med fuld information og uden tilfældige begivenheder må der findes en vindende strategi for en af spillerne. Denne tilfalder første spiller som en konsekvens af at ingen træk kan være til ulempe.

En generel vindende strategi (dvs for vilkårlig størrelse bræt) kan sandsynligvis ikke findes idet problemtypen at finde en sådan, kan vises at være PSPACE-komplet. For disse problemtyper gælder at de kan løses på polynomiel plads men kræver (så vidt vides) eksponentiel tid i forhold til problemets størrelse.

Det er let at spille Hex men sværere at blive god til det. Dette speciale giver nogle råd på baggrund af den matematiske viden.

Reglerne til Hex kan varieres med henblik på at opnå ny viden om spillet. Således er blandt andre *Y* og *Bridg-It* opstået.

Der foregår til stadighed en tilnærmelse af løsningen på Hex. Nogle forsøger sig med at kortlægge hele spiltræer mens hovedparten af ressourcer sættes ind på algoritmer til søgning i og evaluering af positioner på Hex-brætter.

[1] Den danske titel er en direkte oversættelse af et ordspil på en film af Woody Allen.

Med udgangspunkt i at Hex er PSPACE-komplet argumenteres der for at det, trods indsatsen, ikke vil lykkes at løse Hex. Ikke alene har mange års forskning været forgæves, en løsning vil desuden have en uhåndterlig størrelse.

# Chapter 3

# The History of Hex

This contains a history of Hex, from its first invention in 1942 and bringing it right up to the latest research. Beginning with a brief outline I shall proceed to describe in more detail various points of interest.

According to Hein's personal notes he invented Hex while he was pondering the Four-Colour Conjecture, apparently trying to disprove it. I shall describe Hein's ideas in some detail in section 3.1.

In 1942 Piet Hein was asked to lecture in an association for mathematics and natural science students at The University of Copenhagen. The chairman of the association was Aage Bohr (later to recieve the Physics Nobel Prize) whose father Niels (also a Nobel Laureate) knew Hein. They had also invited the student Chess club as the title of the lecture was *Mathematics regarded as Games—The Mathematics of Games*. During his lecture Hein spoke mainly of conditions for good games. He concluded the session with a demonstration of a board game that he had only just constructed and which, according to his conditions, should be a very good game.

On December 26, 1942 the game was described to the general public for the first time in the Danish newspaper *Politiken* under the name **Polygon**.[15] Hein had made an agreement with the newspaper that he was to write a regular column and the paper would publish the game. In section 3.2 I shall provide a reading of a few of these columns.

In 1949 John Nash was a graduate student at Princeton University. He invented Hein's game independently, as an example of a game in which the first player has a winning strategy that is unknown. The game quickly became popular among the other students.

In 1953 the American games company Parker Brothers discovered and marketed the game. They gave it the catchy name **Hex**, obviously because of the hexagonal tiles. This commercial version is now long out of production, but the name is the one that stuck.

One of Martin Gardner's early columns in *Scientific American* mentioned Hex in the summer of 1957 [11]. Gardner was a friend of Piet Hein and it is likely that Hein himself suggested Hex as a subject for the column. In it Gardner summarizes the history of Hex so far and offers a sketch of Nash's proof of the existence of a winning strategy for the first player.

Apparently Gardner's description of Hex was read in Denmark, and in the fall of 1957 Hex was mentioned in a number of scientific and popular Danish magazines. Hein exploited the attention to have the game produced with a big wooden board by the games company Skjøde Skjern. Incidentally, Hugo Piet Hein, the son of Piet Hein is reintroducing and selling a copy of that board now.

Recent research on Hex is focused primarily on creating a program that can play Hex. The programs grow stronger each year, but as yet none have achieved the playing skill of expert human players. Different approaches have been attempted, and in chapter 8 we shall take a look at the most important programs.

At California Polytechnic State University Kevin O'Gorman is building a database (called OHex) of Hex games extracted from some of the computer programs. This database currently offers hints to the best moves in actual play. This exhaustive approach is the only (known) way of finding the winning strategies on specific board sizes, though O'Gorman rightly does not expect to find any for large boards let alone for all sizes. As it can only use the actual games that are in the database it will sometimes give hopeless responses during play. Yet, with every game added the database improves its play. The OHex project will also be treated in chapter 8.

## 3.1 The Invention of Hex

Usually mathematical inventions or discoveries are comprehensively described and we have ample documentation for the common mathematical knowledge before the breakthrough and thus what led to it. That is usually not the case with border areas like that of mathematical games. Yet, in the case of Hex, there are a few sets of notes, one of them being an undated manuscript by Piet Hein which I believe is from his lecture to *The Parenthesis*, a mathematical association at The University of Copenhagen, on an evening in 1942. I found this in Piet Hein's own archives. A translation of it and a few other manuscripts are found in appendix B.2.

During this lecture Hein describes how he came up with Hex. For a number of years he had been examining and creating games as a hobby and managed to set up a list of conditions for good games. This amounts to a list of six conditions: a game must be *fair*, *progressive*, *final*, *easy to comprehend*, *strategic* and *decisive*.

Hein's offset for the lecture was "mathematics as a game, and [Hex] is a simple example of looking at games as mathematics." Hein's point was that mathematics is different from the empirical sciences, in that it creates models that are manipulated according to rules similar to those of a game—these models are in turn compared to the actual world through empirical methods and thus help us make explanations and predictions.

In a sketch for the first column in *Politiken* Piet Hein describes how the complete game of Hex ocurred to him while working with the Four-Colour Conjecture: "The game builds on the simple geometrical property of a planar surface that two lines within a square each connecting a pair of opposite sides must intersect."

He was considering four countries in a ring, realising that only one pair of opposite countries can share a border across the middle. Had it been possible for both pairs of opposite countries to touch across the middle, then a fifth country encircling and touching the first four would disprove the theorem. This image is clearly equivalent to a complete graph with five vertices ($K_5$) embedded in the plane.

Hein's idea was a game in which two players try to connect their two opposing homelands. He writes in [sec. B.2.2] that "This has not been utilised before though it is such a simple quality"—Cameron Browne agrees to the status of Hex as genuinely innovative and describes only two earlier but unknown connection games.[6]

The objective of the game thus being decided Hein still needed the structure of the boards and the rules of the game. He realised that by "constructing the gameboard from square cells—or triangular cells for that matter—then four or more cells will touch by their corners which will stunt the game."[sec. B.2.2] The hexagonal structure is the most simple and elegant solution and the one that both Hein and Nash settled on.

Hein preferred as few and simple rules as possible: "there is no reason to delimit the rule from the general: that markers can be placed anywhere" and in alternating turns. He concludes that he did not even contribute much to the invention.

> When the two halves of the idea—i.e. the crossed connections and the hexagonal grid—had found each other, not only was the idea conceived but the entire game was executed.[sec. B.2.2]

It seems that Hein was fully aware of the game's potential and that he was one of the first to use the connection aspect for a game as well. He was thorough in his exploration of the game and had both mathematicians and chess experts examine it and ultimately used his platform in *Politiken* for the cause as well.

## 3.2 Politiken

When Piet Hein presented Polygon in *Politiken* he was already a familiar name to the readers. For a few years he had been writing his 'Grooks', small aphorisms or poems on everyday life.

Through the months December 1942 to August 1943 Piet Hein wrote a recurring column on Hex in *Politiken*. The first few days contained analyses of openings, discussions of completed games and he had another active period in late January, but eventually the column offered only a problem and the solution to the previous problem. Appendix B.1 is a list of the *Politiken* columns. One notices how the regularity of the beginning fades into no obvious pattern and stops without further notice. The solution to the last problem (number 49) was never even printed.

Piet Hein was very good at marketing his own inventions and this series of problems and contests was doubtless a part of a marketing ploy. However, as mentioned, he was also using his audience in a search for discoveries with regard to a strategy. He knew that the game, being one of complete information, does have a winning strategy and thus kept encouraging the readers to send him what they believed to be winning strategies. He had expert chess players search for the strategy as well and had a good feeling for the complexity of the game.

In August of 1943 the column stopped (or maybe rather ebbed out) for unknown reasons. Perhaps this was when Piet Hein went to Argentina because of the German occupation of Denmark and Hein being married to a Jew—or perhaps he just lost interest. The latter explanation, however unexciting, is the more likely and would also explain the irregular appearance of the column.

It seems that Denmark forgot about Hex for a number of years and had it not been for the invention at Princeton University we would probably not have known it today.

Below we shall look at excerpts from some of the columns, predominantly from the early period. In particular we shall see some of Piet Hein's problems and prize contests.

### 3.2.1 December 26, 1942

> Would you like to learn Polygon? Piet Hein has constructed a game that can be practised with equal joy by the chess expert and one who is merely able to hold a pencil.

This is how the first published article ever about Hex begins. It describes the game in detail and thoroughly stresses the simplicity of the rules and the complexity that not even experienced chess players have been able to see through.

Hein writes that since two lines inside a square each connecting their pair of opposing sides must intersect, it must be possible to create a game that challenges two players to connect opposing sides so that only one can succeed. He does not mention his inspiration from the Four-Colour Theorem as the audience is ordinary readers. "But it does not work to give the board a quadratic pattern as e.g. a Chess board. For on a board where four or more fields meet in one point, the two competing parts will immediately get into a deadlock with each other. So none of them gets a connection. Therefore one is forced to use a board where at most three cells meet. The solution which is simplest and most regular is the hexagonal pattern."

Having explained the simple rules, the article goes on to explain three possible relations between two cells: Contact [adjacent], angle [bridge] and across—the two first being secure connections and the third one unsecure. He offers the advice that playing around the middle of the board is beneficial, but certainly not necessary. An example of a possible opening play is given.

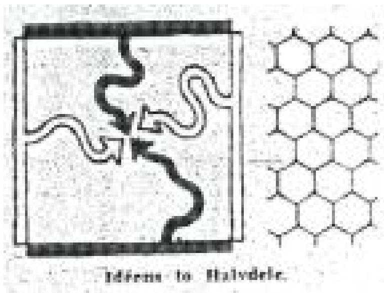


Figure 3.1: The two ideas that led to Hex: Connection and the hexagonal pattern.

Concluding the article is a problem along with the promise that the following days will bring new boards, new opening moves and new problems. There was a prize of 50 Danish kroner for the correct solution to problem 1 in which white is to make a winning move.

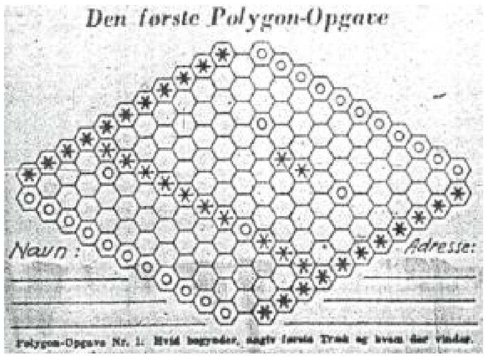



Figure 3.2: Problem 1, The problem that appeared with the first article on Hex on December 26, 1942. White moves once and makes a white win inevitable.

### 3.2.2 December 27, 1942

Hein analyses the three smallest boards ($1 \times 1$ to $3 \times 3$) to make the reader realize some of the fundamental properties of Hex. On the third board, Hein says, it is easy to find both winning and losing openings, but two winning openings will result in the board being filled completely, provided that both players play optimally. [These are the middle cells along the opponent's side.]

On a board of $4 \times 4$ cells Hein suggests a game variant with a starting position in which White occupies the two acute corners and Black one of the middle cells. With white player moving next he can win, but not that easily. The fifth board has potential for good play and difficult problems, Hein says, and one such is problem 2.

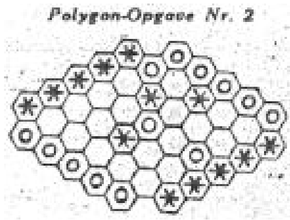



Figure 3.3: Problem 2, White to move and win. *Politiken*, December 27, 1942.

### 3.2.3 December 28, 1942

This column describes how a game might begin. In figure 3.4 we notice a very close opening play resulting in the best position for black with bold numbers.

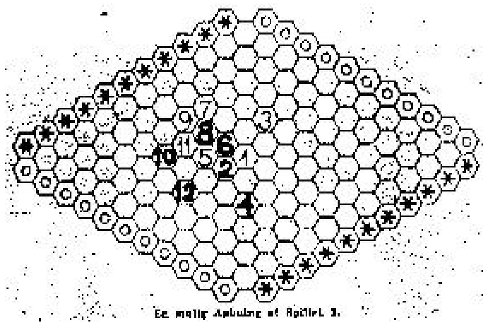


Figure 3.4: A possible opening of Hex, *Politiken* December 28, 1942.

Once more we have a problem, this time on a 6x6 board. As usual it is white to move and win.

### 3.2.4 January 1, 1943

New Year's Day was the day that *Politiken* announced the winner of the competition from December 26 and also arranged a new one. The new competition consisted simply in playing a game of Hex, enumerating the

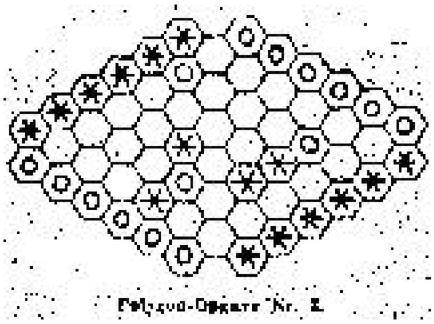


Figure 3.5: Problem 3, White to move and win. Politiken, December 28, 1942.

moves and sending it to *Politiken.* 100 Danish kroner would be awarded to the best game—not stating any criteria.

In order to encourage new players the article states that: "Anyone can play Polygon. You merely hit upon an empty cell and put your mark there. It can be done with various degrees of talent, sure, but do you know if you do not possess that particular ability?"

### 3.2.5 January 9, 1943

Apparently *Politiken* received quite a lot of mail regarding Polygon. Many of the letters described what the senders believed were sure-fire strategies for the first player. They would all open on the middle cell and advance towards the sides by moving in a straight line either horizontally or perpendicular to the side.

Hein describes a counter-offensive to this strategy which involves close play eventually forcing a ladder [more on these terms later]—and in case someone still has a sure-fire strategy he wishes to see it. The counter-offensive is illustrated in figure 3.6.

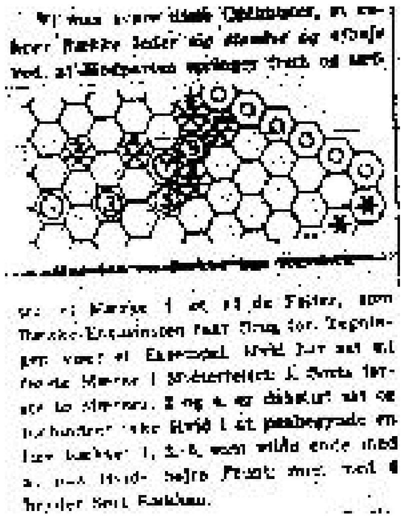


Figure 3.6: Counter-offensive against an advancing series of bridges: "*any* row can be blocked."

### 3.2.6 January 17, 1943

Some days only featured an advertisement for Hex as this one picturing a couple absorbed in playing Hex in an air-raid shelter and a woman from the civil defence telling


Figure 3.7: An advertisement from *Politiken* January 17, 1943. A woman from the civil defence sticks her head into an air-raid shelter saying: “Ladies and Gentlemen, the All-Clear was sounded!!” Playing pads with fifty leaves were sold in bookstores.

them that the All-Clear was sounded. Thus, this advertisement is really a little piece of documentation of life during the Occupation of Denmark.

The text in the middle begins: "Yes, one quite forgets one's surroundings—however disturbing they are—when one absorbes oneself in a game of POLYGON."

Politiken had made pads with hex boards and sold them through bookstores all over Denmark. "There are 50 games in each pad, enough for a party or a night in the air-raid shelter - should it come to that!"

### 3.2.7 January 20, 1943

The winner of the open prize contest was a student whose game is depicted in figure 3.8. The column says that after a first rough grading there were still more than a hundred good games to choose from.

The winning game is annotated and exhibits excellent play, including most of the advice given in chapter 6.

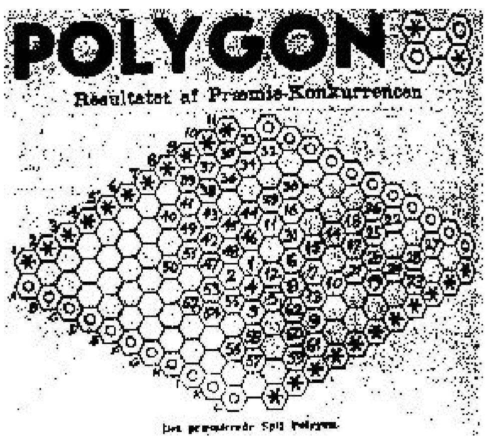



Figure 3.8: The game that was declared the best game and won its participants 100 Danish kroner. Printed on January 20, 1943.

## 3.3 Princeton University

As mentioned above John Nash was a mathematics student at Princeton University in the late forties. In the common room of "Fine Hall" (the Mathematics Department building) students and teachers would meet between classes for learned discussions over a cup of tea. In a personal e-mail Nash recollects that they played Go a lot and also some Chess and other games. This informal environment inspired him to develop an outline of Hex as an example of a game with a winning strategy for the first player.

An older student, David Gale was among the first to see the game and realised at once the entertainment value of the game. He created a board that was available in the common room. In a personal e-mail Gale offers his own description:

> One morning Nash told me he had an example of a game which he could prove was a first player win but could see no way of finding winning strategies. He then described the checker board version. I was intrigued and thought it was not only theoretically interesting but might be fun to play. I spent most of the rest of the day making the hexagonally paved board which then lived for many years in the Fine Hall common room and became quite popular.

The checker board version is Hex played on the vertices of a checker board with diagonals in one direction added.

In his honour the game was known as Nash at Fine Hall. A popular but probably apocryphal story claims that the game was played on the tiles of the dormitory lavatories and therefore nicknamed 'John'.

The game spread, at least to Yale University. Anatole Beck mentions becoming familiar with the game before the Parker Brothers' version was released.[2]

Gale also tells how a few years after the Parker Brothers' game was marketed he received a call from Nash who accused him (wrongly) of double crossing and having sold the Parker Brothers the game. How the Parker Brothers actually discovered the game remains unknown.

## 3.4 Biography: Piet Hein

Piet Hein was born on December 16, 1905 (so this year he would have been a hundred!) in Denmark. He was from a family of scholars and related to the famous author Karen Blixen. Scientists and artists frequently visited his family and the author Johannes V. Jensen is said to have inspired him to write poetry.

After attending Metropolitanskolen, a prominent upper secondary school in Copenhagen, he went to Sweden to become a painter at The Royal University College of Fine Arts. Before completing his studies he went back to Copenhagen to study theoretical physics under Niels Bohr who was also a friend of his family.

He did not graduate from The University of Copenhagen either, but commenced inventing technical wonders and games, writing poetry and agitating for intercultural understanding and cooperation.

During the German occupation of Denmark 1940–1945 Piet Hein was affiliated with the Danish newspaper *Politiken*. He mainly wrote small poems, in time developing a distinct style which he called 'Grooks'. At least in Denmark, these are what Hein is best known for.

Hein also contributed to architecture and design. He is particularly known for his use of the Super Ellipse to solve a traffic problem in Stockholm.

A number of prizes were awarded to Hein, including The Alexander Graham Bell Silver Bell in 1968 and an honorary doctorate at Yale University in 1972 and Odense University (now University of Southern Denmark) in 1991.

Piet Hein died on April 17, 1996, ninety years old.

## 3.5 Biography: John Forbes Nash

This biography is mainly based on Nasar and Nash in *The Essential John Nash.*[16]

Nash was born in West Virginia, USA on June 13, 1928. His father, John Nash senior was an electrical engineer and provided him with an encyclopedia and gadgets to stimulate his scientific interest.

As a young schoolboy he showed great prowess in mathematics and chemistry and for a while actually studied chemistry but changed to mathematics towards the end of his undergraduate study. In 1948 at the age of twenty, Nash was accepted into Princeton University as a graduate student of mathematics with very good recommendations.

It took Nash only a little over a year to finish his 27 pages Ph.D. thesis on non-cooperative games which was later to give him the Nobel Prize.

After receiving his degree in 1950 Nash went to Massachusetts Institute of Technology to teach. Having married and expecting his second son, Nash fell victim to a serious case of schizophrenia forcing him to leave his job and a promising academic career for around 25 years.

In the meantime, his contributions to both game theory and geometry had increased in significance and in 1994 he was awarded the Nobel Prize in economics, with John C. Harsanyi and Reinhard Selten. Nash's mental condition had been improving, but the recognition apparently gave him a final push back into 'the real world'. Nash resumed his research at Princeton and has since been working on a mathematical model for the expansion of the universe as well as game theory and geometry.

# Chapter 4

# Game Theory

Hex is an interesting game mathematically. That is the reason for beginning this project in the first place. This is a central chapter that will make explicit the conflict that keeps attracting mathematicians and the like.

Hex belongs to the category of finite two-player games, with no chance events and complete information. In the same category we find Go, which was played excessively at Princeton in the late forties, but also Bridg-It, invented by David Gale, which is quite similar to Hex but solved in the general case. Chess also belongs here (if we include the clause that a stalemate equals a draw).

In any finite two-player game with complete information and no chance events (deterministic) there is an optimum strategy.[7] The strategy consists in maximising respectively minimising the values assigned to each game state. The value $v$ of game state $T_i$ is assigned recursively ($T_i$ is followed by one or more possible $T_{i+1}$):

- $v(T_i) = 1$ if player A has won
- $v(T_i) = -1$ if player B has won
- $v(T_i) = \max(v(T_{i+1}))$ if player A to move
- $v(T_i) = \min(v(T_{i+1}))$ if player B to move

With this assignment of values, obviously player B must minimise on every move and player A maximise. In many deterministic games the result of the optimum strategy for a player is a loss. As we shall see in this chapter the player to move second cannot obtain anything better if the first player follows his optimum strategy.

Knowing the optimum strategy potentially ruins a game. We still play them, though, because for large games this assignment of values is only possible in principle and as we shall see, the actual winning strategy in Hex remains hidden.

Nash used this knowledge of general game theory when designing Hex as a game with a winning strategy for the first player. We shall see his outline of a proof of the property made explicit in this chapter.

## 4.1 The existence of a winning strategy

It was probably discovered by several mathematicians independently that there is a winning strategy and that it belongs to the first player. According to Gardner, Nash outlined a proof in 1949 and Hein's own writings suggest that he also knew this.[11] Hein's implicit argument is that since the players' connections cannot both be blocked locally (in a triple of cells), they cannot both be blocked globally either.[15, 42-12-26] It seems, however, that a crucial part of the proof was not published before 1969 by Anatole Beck in [2], namely the no-draw property of the Hex board.

This section deals with Nash's sketch proof as printed in [11] and develops certain aspects further:

1. Either the first or second player must win, therefore there must be a winning strategy for either the first or second player.

2. Let us assume that the second player has a winning strategy.

3. The first player can now adopt the following defense. He first makes an arbitrary move. Thereafter he plays the winning second-player strategy assumed above. In short, he becomes the second player, but with an extra piece placed somewhere on the board. [This argument is usually referred to as 'strategy stealing'.]

4. This extra piece cannot interfere with the first player's imitation of the winning strategy, for an extra piece is always an asset and never a handicap. Therefore the first player can win.

5. Since we have now contradicted our assumption that there is a winning strategy for the second player, we are forced to drop this assumption.

6. Consequently there must be a winning strategy for the first player.

In the following I shall expand these arguments and give formal proof where necessary. 1. is treated in the section immediately below, 3. and 4. in the next. The rest is a simple logical exercise.

### 4.1.1 One winner

*Ad. 1.*

It is a key property of Hex that there will always be exactly one winner as I shall prove here. We will initially state the fact in the most general form as a lemma and from that deduce the wanted.

**Lemma 1** *A Hex game cannot end in a draw.*

In order to prove this lemma, it is necessary to realise that any game has at most three possible outcomes:

- Either one player wins and the other does not;
- or both players win;
- or none of the players win.

It is a matter of definition specific to the game what the difference between the second and the third outcome is—often there is none. In Hex it is quite straight-forward that a win for a player equals having established a chain connecting his two sides. Thus 'both players win' and 'no player wins' translates as 'both players connect their sides' and 'no player connects his sides'.

Lemma 1 is equivalent to the statement that no valid distribution of pieces on a Hex board contains connecting chains in both directions and all valid completed boards contain at least one connecting chain. Valid means any distribution that can be obtained during play, but in fact any distribution has this property.

It is easy to convince oneself of the property—being a consequence of the impossibility of a planar complete graph with five vertices and the Four-Colour Theorem—but a proof is less trivial. This proof is a version similar to the one in [10] but building more on graph theory.

**Proof** The proof has two parts: The players cannot both win and the players cannot both lose.

The players cannot both win since, as soon as one of the players wins, i.e. completes the necessary connecting chain, the game ends and the other player will not be allowed another move to complete his connection.

For the other part of the proof we shall consider the Hex board as a planar graph, each vertex being a point in which the corners of two or three tiles meet, and all being connected in the same way as on the board. The degree of all vertices is $\leq 3$. Notice how, along the sides of the board, the hexagonal structure is scrambled. The corners are considered to constitute vertices as well because this is where the players' homelands meet.

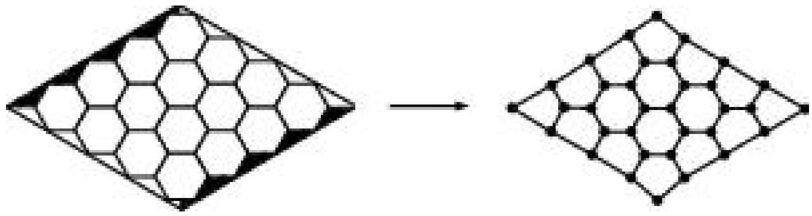


Figure 4.1: A $4 \times 4$ Hex board seen as the planar graph $G$.

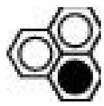


Figure 4.2: Each triple of cells converts to two edges in $G'$—or none.

Let $G$ be the planar 2-connected graph with all vertices of degree 2 or 3 (outer and inner vertices) and consider all cells of the original Hex board occupied as in a finished game. We can now create a subgraph $G'$ of $G$ by colouring all edges lying between two differently occupied tiles on the Hex board.

All inner vertices in $G$ will be of degree 0 or 2 in $G'$. *Either* the three surrounding cells share the same colour *or* one of the three has a different colour as in Figure 4.2. In the first case the vertex will be of degree 0, in the second 2. Most of the outer vertices in $G$ will have degree 0 as their surrounding areas are of the same colour—only the four vertices at the corners will be of degree 1. This is true for any size of the board (and corresponding graph) and for any distribution of pieces.

Now $G'$ is a subgraph of $G$ with all vertices of degree $\leq 2$. It is a well-established fact of graph theory (*König's theorem*) that $G'$ will consist purely of disjoint simple cycles, simple paths and isolated points. Since we have exactly four vertices of degree 1 there must be exactly two simple paths and as they cannot intersect in a planar graph, these must each connect one of *East/West* to one of *North/South* (and not the same).

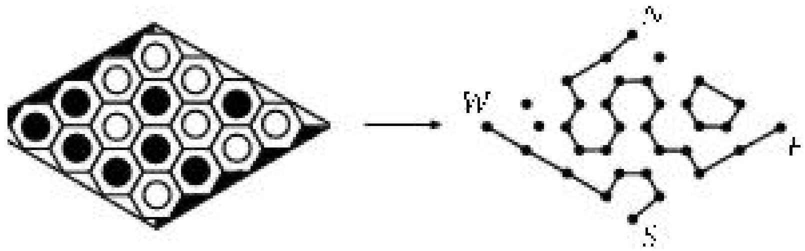


Figure 4.3: $G'$ is a subgraph of $G$ created by colouring all edges lying between different colours on our corresponding Hex board.

More formally, if the two lines connect *North* to *South* and *East* to *West* then along with the sides of the board and a fifth vertex, added outside the board and connected to each of the corners, they will constitute a planar version of $K_5$. This is where Hein realised that both players succeeding in connecting their homelands would imply a disproof of the Four-Colour Conjecture.

Assume that *West* is connected to *South* as is the case in figure 4.3. Following the line from *West* to *South* will define a chain on the left-hand side connecting the two sides belonging to black. Since the paths cannot intersect, and thus not touch, the minimum connecting chain will lie between the two paths. This will also be the case when *West* connects to *North*. □

The proof can be generalised to account for any board as long as its graph representation is a 2-connected graph with all vertices of degree $\leq 3$. This proves Hein to be right in using impossible local blocking as an argument for impossible global blocking.

I demonstrated quickly and easily that both players cannot win, exploiting a rule that is entirely contingent. In fact the lemma would also be true in a game that only finishes when the board is full. This also follows easily from the necessary properties of the subgraph $G'$ examined above. A simple path between two corners effectively blocks all possible connections.

Now, considering that any game of Hex is played on a board with $n \times n$ cells, keeping in mind that no pieces can be removed, it is quite trivial that all games will consist of at most $n^2$ moves. Since we have just shown that in the case of a completed board there must be exactly one winner this will apply to any finished game—as a game does not finish before a winning chain is established. This leads to the formulation of the following corollary:

**Corollary 2** *Either the first or second player must win, but not both.*

The property that this corollary proves was one of Hein's characteristics of a good game as described in section 3.1. The assumption that $G$ contains no vertex of degree $> 3$ is necessary and makes it clear why Hex cannot be played (satisfactorily) on a squared board.

**Remark** We have now established that exactly one of the players will win any game of Hex. From this fact follows the existence of a winning strategy, keeping in mind that Hex is a game of complete information and no chance events, in principle fully determinable. This is the implicit statement of 1. in Nash's proof above.

### The Brouwer Fixed-Point Theorem

The lemma and corollary just proven appear to have mathematical consequences. In *The Game of Hex and the Brouwer Fixed-Point Theorem* David

Gale has shown the property of Hex to be equivalent to a fundamental theorem of algebraic topology, namely the Brouwer Fixed-Point Theorem.[10] I shall not examine this fact in depth but merely remind of its content.

The Brouwer Fixed-Point Theorem says that for any continuous mapping $f\colon I^2 \to I^2$ from the unit square onto itself there exists an $x \in I^2$ such that $f(x) = x$.

The theorem also comes in a generalised form and by equivalence Hex must too. However, a generalisation of Hex to more dimensions and more players will result in a poor game as it will be extremely difficult to block each other—more so for every extra dimension.

### 4.1.2 The strategy stealing argument

*Ad. 3.*

The strategy stealing argument was developed for Hex by John Nash. It is obviously applicable to many symmetric two-player games but must be accompanied by the rest of the arguments above in order to prove anything at all about the existence of a winning strategy.

The argument is that if the second player has a winning strategy $S$, so will the first player. He can play his first move arbitrarily and in the rest of the game pretend he is the second player—assuming $S$. In case this strategy requires occupying the cell that was occupied in the first arbitrary move, the player will make another arbitrary move and continue along the lines of $S$. The same applies to subsequent cases of cells being already occupied.

*Ad. 4.*

A necessary property for the validity of the strategy stealing argument, is that a piece can never be a disadvantage. This property is intrinsic to Hex as a game of laying out stepping stones between two shores. In Hex only the best (minimal) connection, finished or unfinished, between a player's two sides plays a role. An 'extra' piece can be added and make this connection shorter in which case the piece is an advantage; or the piece can make no difference to the number of pieces in the minimal connection, in which case it is not a disadvantage. An extra piece can never be disadvantageous.

It is clear why the second player cannot simply steal the first player's winning strategy: In case the cell is already occupied by the first move, which he ignored in order to pretend to be the first player, the strategy fails because it is not his piece.

Considering that a winning strategy offers exactly a guaranteed win, the strategy stealing argument makes a contradiction to corollary 2 by implying that both players must have a winning strategy and the assumption that the second player has a winning strategy must be false.

### 4.1.3 First player wins

*Ad. 6.*

Having made explicit the arguments of the proof above we can now state in a formal way the following:

**Theorem 3** *For any $n \times n$ game of Hex the first player has a winning strategy.*

We notice (regretfully) that the theorem is only one of existence and makes no promise of a practical construction of the winning strategy. In the next chapter we shall see that there is very little hope of discovering a winning strategy in the general case. We can only hope to develop winning strategies for small boards—so far this has been accomplished for boards up to $9 \times 9$.

# Chapter 5

# The Complexity of Hex

We have just seen above that Hex does have a winning strategy for the first player on any board size, but that we have no idea what it looks like. We can say further that we cannot even expect to discover it. In this section I shall present further evidence for this discouraging statement.

We are looking for a general winning strategy, i.e. a strategy applicable to any board size. But even finding a winning strategy for one board is a tremendous task. We begin this chapter by doing some calculations as to how long time it would take to find a solution to some of the smaller boards.

The second and major part of the chapter deals with complexity theory and the categorisation of Hex in the hierarchy of hard problems. This theory gives us reason to believe that we will never discover a general winning strategy.

## 5.1 Solving Hex with brute force

There is a solution to Hex and in principle it is directly accessible to us. One method for finding a winning strategy is in fact quite straight-forward: Sit down and start playing all possible games. Using some automation, this should not be too hard, should it? Indeed it is. Let us try and do some calculations as to how long it would take.

Being a quite systematic person, I would begin on an $n \times n$ board by numbering the cells from 1 to $n^2$ and playing them in numerical order, the first complete game being $[1, 2, 3, \ldots, n^2 - 2, n^2 - 1, n^2]$ and the next $[1, 2, 3, \ldots, n^2 - 2, n^2, n^2 - 1]$ and so on. This is easily seen to create $n^2!$ games to play.

However, we can ignore quite a lot of games:

1. The games that begin $[1, 2, 3, 4, 5 \ldots]$ are the same as those that begin $[3, 2, 1, 4, 5 \ldots]$.

2. Quite a lot of the games would finish before the board was full and we

can ignore the rest of the moves in these. A rough (and optimistic) estimate would be that an average game uses half the available cells.

3. Finally, half the games are covered by rotation.

These reductions amount to a total number of games that is the number of ways one player can occupy a quarter of the cells times the number of ways the other player can occupy another quarter of cells divided by two:

$$\frac{\binom{n^2}{\frac{1}{4}n^2} \cdot \binom{\frac{3}{4}n^2}{\frac{1}{4}n^2}}{2}$$

If I work efficiently and ignore interruptive bodily needs, I can make one move every second of the day. Each game will take in average $n^2/2$ seconds to play and finding the full solution for a board of size $n \times n$ thus takes

$$\frac{n^2! \cdot \left(\frac{3}{4}n^2\right)! \cdot n^2}{\left(\frac{1}{4}n^2\right)! \cdot \left(\frac{3}{4}n^2\right)! \cdot \left(\frac{1}{4}n^2\right)! \cdot \left(\frac{1}{2}n^2\right)! \cdot 4}$$

seconds.

Solving the $2 \times 2$ board will take 12 seconds. $3 \times 3$ should be about 2,400 seconds. These have already been solved, so I would much rather solve the standard board of $11 \times 11$ cells. I should be able to do this in only $10^{54}$ seconds—or $3 \cdot 10^{46}$ years. In comparison, our solar system is believed to have existed in 4.5 billion years.

Obviously, my calculations were to be done by hand and a computer is *much* faster than I am—and they can in fact even work all day and night. So let us cut down my estimates with a factor of one million or even a billion. No matter how many of today's computers we combine, they will not be able to solve standard size Hex in a reasonable period of time.

These calculations deal with only specific sizes of Hex and offer no hints to a *general* solution. Solving the next size would take considerably longer and apparently, there is no upper limit to how long it would take to solve all boards using this approach.

## 5.2 Complexity

When talking about complexity in mathematics or just problem-solving in general, there is always the first question of whether or not a solution to the problem we are discussing ultimately does exist. In many cases, when a solution is not obvious, it is possible to prove that there *are* solutions, even if they may be extremely difficult to find. In other cases we can prove problems undecidable, and of course there are those that constantly avoid classification as decidable or undecidable.

An important distinction is between truly undecidable problems and those that are only “intractable”. According to Garey and Johnson, Alan Turing was one of the first people to prove a problem type undecidable.[13] In the thirties he showed that it is truly undecidable for a general algorithm and input to it whether or not the algorithm in question will run indefinitely or it will halt after a finite number of steps. In order to make this a precise mathematical theorem, *general algorithm* is to be interpreted as an abstract Turing machine as defined by Alan Turing, constituting the standard model for the concept ‘algorithm’.

Hex, being a finite $n \times n$ game, is in principle solvable, as shown in the previous section, but is very likely intractable. Intractable means that a problem type cannot possibly be solved by any algorithm using only polynomial time. Hex has been partly solved for boards up to $9 \times 9$ but in this context we are only truly interested in a general solution. Solved means that for any given board position it is known which one of the players has a winning strategy—and phrasing the problem type properly, the solution will also reveal the winning strategy. Thus a solution is an algorithm (running in polynomial time) that will reveal a winning move for any board size and position.

We shall now differentiate our concept of intractability slightly by introducing different complexity classes, each describing computational needs for solving their member problem types. When solving problems there are ultimately two resources that are in limited stock: time and space.

In the following I shall discuss briefly a small part of complexity theory, hopefully without things becoming too technical. An important feature of this theory is that problems and questions are usually rephrased into decision problems, i.e. questions that can be answered with a ‘yes’ or a ‘no’. This makes it possible to transform problems into each other and compare them.

In order to make clear what question we are trying to solve (or rather, trying to show intractable) let us first phrase Hex explicitly as a problem type:

| Problem type | Hex |
|---|---|
| Given | A partially filled (possibly empty) $n \times n$ Hex-board and an empty position $p$. |
| Question | Is playing at $p$ part of any winning strategy? |

The question may be asked for the player in turn. In case of a ‘no’ the player should ask the question again with a new position until all empty cells have resulted in negative answers. In that case, clearly, the opponent has a winning strategy regardless of the player’s move.

It is possible to answer the question—and in turn to discover a winning strategy—for any board size in finite time. However, the time required for a solution to a given board seems to grow more than polynomially with the problem size (i.e. board size).

In the rest of this chapter, we shall describe the complexity theory to make probable that our problem is intractable — and prove it to be at least as difficult as many other hard (yet unsolved) problems.

### 5.2.1 Classes P and NP

In section 5.1 we made some calculations showing that an overwhelming amount of time is needed in order to solve just the standard version of Hex with the only known approach, which is by exhaustion. The amount grows exponentially with the board size. Thus, time is really an issue here.

In the 1960s, J. Edmonds introduced the notion of a *good* algorithm as being one that produces an answer in at most polynomial time, relative to the problem size.[13] For Hex the problem size is the number $n$ of cells on the board—usually we denote the number of cells along the edge by $n$ but for computational purposes we prefer the complete number of cells; any confusion of the two makes no practical difference anyhow since being bound by a polynomial in $n^2$ is equivalent to being bound by a polynomial in $n$. In the following, unless otherwise stated, when I mention a solution to Hex, in fact I mean a good solution, i.e. a polynomial time algorithm that answers our question about a winning strategy for the player in turn.

Along with this definition follows a complexity class P (for **p**olynomial) of problem types which are solvable with *good* algorithms. For the time being, no good algorithm is known to solve Hex, and the same goes for a long list of other problem types.

Sometimes it is possible to guess a solution to a hard problem—or it might even be revealed in a vision or dream—and in some of these cases the solution will be verifiable in polynomial time. For this reason we use NP (for **n**on-deterministic **p**olynomial, non-deterministic referring to the source of the certificate) to denote the complexity class of problem types that, in case of a positive answer, have a certificate that we can verify with a good algorithm.

A certificate is a guess at a specific strategy that proves the answer to the question to be 'yes'. It must be phrased as new input and an algorithm describing what to do. Apparently, in order to verify a strategy in Hex we will need to confront it with all possible opposition (which exceeds polynomiality).

Problem types in P are, for obvious reasons, easily verifiable, even with an empty certificate, and thus all members of P belong to NP. It is possible that the existence of a good algorithm for verifying certificates implies the existence of a good algorithm for solving a problem, but it is as yet undecided whether P = NP or not. In fact, there is currently a prize of $1,000,000 for a proof of the equality or its negation. General belief is that P is a proper subset of NP.

A consequence of the lacking answer to the question P = NP is that there are problems in NP to which we in fact do not know whether a good solution exists or not. One thing we do know about a certain group of these, however, is that *if* there is a polynomial solution to just one of them, then we can find polynomial solutions to all problems in NP. This family is called the NP-complete problems. A problem is determined to be NP-complete if it is possible to reduce all problems in NP to it with a polynomial transformation—which implies that transforming an already known NP-complete problem into a problem in NP will prove it NP-complete.

It has been shown that Hex is *at least* as hard as any NP-complete problem, for which we use the term NP-hard. If Hex *is* in NP, it will thus be NP-complete. In the following section we shall discover a class of problems that are probably harder than all NP-complete problems; a class in which we also find Hex.

### 5.2.2 PSPACE

Now, with the knowledge available at the moment, it is impossible to solve Hex for large board sizes because there simply is not enough time available. As mentioned above, another limited resource is space. This is to be understood quite literally as the space taken up by computer memory or even writing paper. Since both humans and computers can only manage simple calculations, we will have to write down intermediate results.

According to Kevin O'Gorman's calculations, an extremely efficient storage of the game tree for a $10 \times 10$ board will require a computer of 1 $km^3$, not considering wiring and cooling.[http://hex.kosmanor.com/hex-bin/notes.html] This huge number comes from an optimistic estimate of the number of possible board positions to be $10^{39}$, and storing only one bit per position in a cube makes it $10^{13}$ bits along each edge. Now, utilising the smallest imaginable storing, non-moving, stacked hydrogen atoms, each about $10^{-10}$ metres; if each atom carries one bit of information we end up with this giant cube of 1 $km^3$. And this was only the game tree of one specific (and small) board size, mind you.

We can, however, solve Hex without storing the entire game tree at once. With a simple minimax search (which will be described in more detail later) we can assign each node, representing a board position, a value according to which one of the players will win, only storing the calculated values closest to the root and thus we only need a few more placeholders than the maximum depth of the game tree, equal to the maximum number of moves $n^2$ in a game. The class of problem types solvable with only limited (i.e. polynomial) storage need is called PSPACE for obvious reasons.

Problem types in both P and NP must necessarily belong to PSPACE, since even if we store all steps of a polynomial time algorithm at once, we cannot use more than polynomial space in polynomial time. Thus follows the relation P $\subseteq$ NP $\subseteq$ PSPACE and it is strongly believed, though not proven, that both inclusions are proper.

The characterization of Hex as NP-hard can be taken a step further; Hex is actually PSPACE-complete and from this follows that if Hex also belongs to NP or P then NP = PSPACE respectively P = PSPACE. This does not prove that Hex is in fact intractable, but it shows that all futile efforts to solve other PSPACE-complete problems were also implicitly directed at Hex and makes intractability quite likely.

### 5.2.3 Hex is PSPACE-complete

Stefan Reisch gives a proof of the PSPACE-completeness of Hex in his article *Hex ist PSPACE-vollständig.*[19] I shall not reproduce it here but give a short outline along with a few examples.

A few years earlier Simon Even and R. E. Tarjan showed in [9] that a generalised version of Hex, known as The Shannon Switching Game played on vertices, is PSPACE-complete. This proof is usually the only one mentioned in relation to Hex—because the more specific proof of Reisch is only available in German, I assume.

The general method for proving any problem to belong to a certain complexity class is to transform another problem from the class to the problem in question using a polynomial transformation and so that a solution to the original problem is directly translatable to a solution to the other. QBF (quantified Boolean formula) is a fundamental PSPACE-complete problem which is often used as basis for these transformations.

Reisch's proof establishes a method for the transformation of any QBF to an equivalent situation in Hex, which can be carried out in polynomial time via a number of simple transformations. This proves Hex to be PSPACE-hard and along with the knowledge from above, that Hex belongs to PSPACE, it must also be PSPACE-complete.

A quantified Boolean formula is a logical expression of the form $Q_1x_1Q_2x_2\dots Q_mx_mF$ in which $F$ is a well-formed formula in conjunctive normal form, $x_i$ is a Boolean variable and $Q_i \in \{\exists, \forall\}$ is an existential or a universal quantifier. A QBF may for example be the following:

$$\exists x_1 \forall x_2 \exists x_3 : (x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3).$$

The formula is in standard logical notation and reads: "there *exists* a value of $x_1$ such that for *all* values of $x_2$ there *exists* a value of $x_3$ such that either $x_1$ or $x_2$ is true *and* either $x_2$ or $x_3$ is true *and* either the negation of $x_2$ or the negation of $x_3$ is true". There are several assignments of truth values to these variables that make the formula true, e.g. $x_1 = x_2 =$ true and $x_3 =$ false.

Reisch transforms a general QBF to Hex via two intermediate problem types in the form of games following this schedule:

1. QBF $\longrightarrow$ Geography
2. Geography $\longrightarrow$ Graph-Hex
3. Graph-Hex $\longrightarrow$ Hex

Geography is originally a game in which players alternately mention names of locations with the only rule that a new location must begin with the letter by which the previous location ended. The game may be formulated as a two-player game on a directed graph in which the players alternately colour a vertex connected to the last coloured vertex by an edge; the player who is not able to colour a vertex loses. The formal problem type is this:

| Problem type | Geography |
|---|---|
| Given | A directed, partially coloured graph and a vertex $v$ directly connected to the last coloured vertex. |
| Question | Is playing at $v$ part of any winning strategy? |

Graph-Hex is equivalent to the Hex we know, only it is played on the vertices of an undirected, planar graph with two distinct vertices $s$ and $t$. In order to win, the white player must connect the two by a path containing only vertices coloured in his colour—black player wins if he prevents white player from doing so. The players alternately colour one uncoloured vertex.

| Problem type | Graph-Hex |
|---|---|
| Given | An undirected, planar, partially coloured Graph-Hex graph and an uncoloured vertex $v$. |
| Question | Is playing at $v$ part of any winning strategy? |

The resemblance of the problem types described is great and the object of the proof is to show that a solution to one is directly translatable to another.

**QBF $\longrightarrow$ Geography:** The QBF's consist of two parts, namely the quantified variables and the clauses. We create a graph from these by connecting the figures 5.1 according to the quantifier of each variable. The left diamond corresponds to an existential quantifier and the right one to a universal quantifier.

The Geography graph is constituted by a starting vertex $s$, a chain of 'variable diamonds' and a vertex $t$. To $t$ we attach a vertex $y_i$ for each disjunctive clause of the formula and further connect these to the vertices representing the opposite truth assignments of the variables in the clauses.

In the cases where a vertex has degree four or more, we replace it by a subgraph so that every vertex has degree $\leq 3$ and yet the game properties of the full graph remain unchanged.

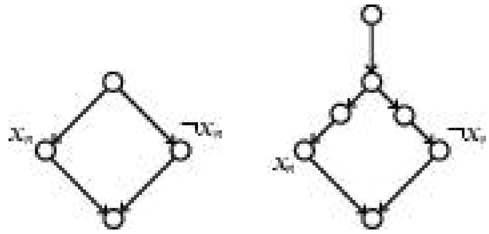


Figure 5.1: Geography subgraphs corresponding to an existentially quantified variable and a universally quantified variable in a QBF. The named vertices represent the possible truth values.

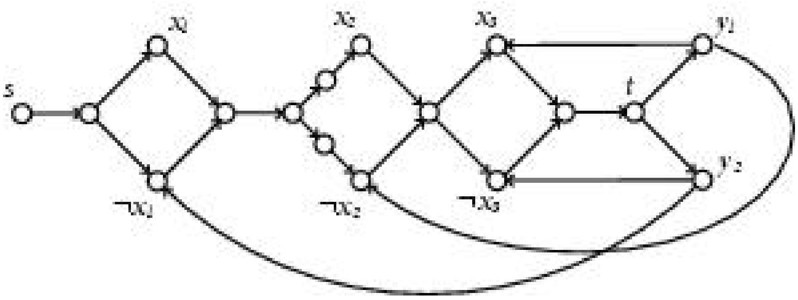



Figure 5.2: $\exists x_1 \forall x_2 \exists x_3 : (x_2 \vee \neg x_3) \wedge (x_1 \vee x_3)$.

The result is a bipartite directed graph with all vertices of degree $\leq 3$ that is almost planar. Only edges going from a $y_i$ disrupt the planarity and we insert subgraphs at intersections that preserve the relationship with the QBF but render the Geography graph planar.

The resulting game graph has a winning strategy for the first player if and only if there is a truth assignment to the variables of the QBF that makes it true. In the example of figure 5.2 $x_1$ = true and $x_3$ = false will make the QBF true and also indicates a winning strategy for the corresponding Geography game, namely for the first player to visit both $x_1$ and $\neg x_3$ on the passage from left to right.

The transformation uses little more than one step for each variable and disjunctive clause and so can be carried out in at most polynomial time, implying that since QBF is PSPACE-complete, determining a winner in Geography is PSPACE-hard.

**Geography $\longrightarrow$ Graph-Hex:** Graph-Hex is played on a graph with two distinct vertices, $s$ and $t$. White plays to connect the two and black to prevent it. The graph is planar and undirected and we must of course preserve the winning strategy from the Geography game for the success of the proof.

Again, a small number of distinct subgraphs are defined and the vertices of the Geography graph are replaced by these according to the number of incoming and outgoing edges. Most of the vertices $v$ of the Geography graph have degree 3, that is either $indeg(v) = 1, outdeg(v) = 2$ or $indeg(v) = 2, outdeg(v) = 1$. These vertices are replaced by elementary graphs according to whether $v$ 'belongs' to white or black player (the player who can move at it).

Otherwise the vertices have $indeg(v) = outdeg(v) = 1$ and thus have no other effect on the game than to shift the initiative from one player to the other determining who gets to choose direction on a following vertex of $outdeg(v) = 2$. In Graph-Hex, these will be ignored since vertices can be selected arbitrarily in this game. Only $s$ and $t$ will remain unchanged.

I will not go into detail about the subgraphs except show one of them. Figure 5.3 is a subgraph of a Graph-Hex graph to represent a vertex in a Geography graph.

Once more, the complete result is an almost planar graph and using somewhat similar replacements as before, we get an undirected, planar graph with all vertices of degree $\leq 3$.

**Graph-Hex $\longrightarrow$ Hex:** Graph-Hex is played on a graph consisting of vertices and edges and with a white player perspective. In transforming it to Hex we replace edges with lines of white stones flanked by black stones. Empty vertices are replaced by empty cells on the Hex board surrounded by occupied cells, whose colour is determined by the degree of the vertex. Occupied vertices are replaced by corresponding cells.

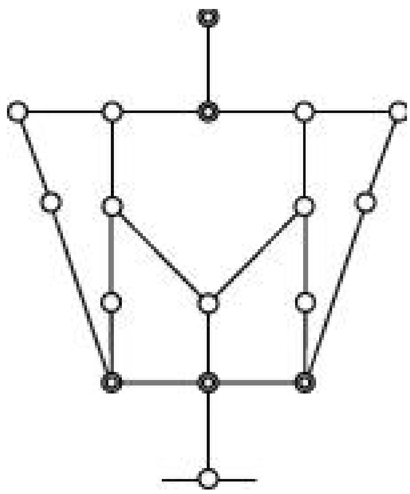


Figure 5.3: A "white choice graph" that replaces a Geography vertex $v$ with $indeg(v) = 1$, $outdeg(v) = 2$ and in which the white player can choose to connect the top to either the right or the left side. The bottom represents a connection to both sides which black can prevent.

This transformation renders a quite large Hex position even when the initial QBF has only few variables and clauses. This is due to the transformation process in which we twice replaced singletons by subgraphs.

The constructive method shown here proves that any QBF can be transformed into a Hex position. Thus, solving Hex in general is *at least* as hard as solving QBF in general and so Hex is PSPACE-hard. With Hex

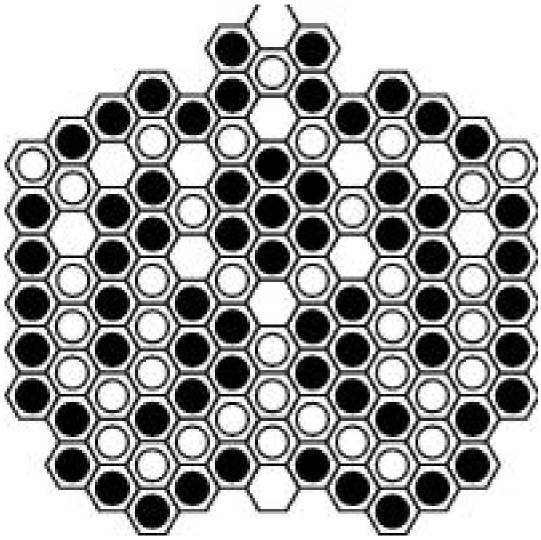


Figure 5.4: The transformation of the white choice graph in figure 5.3 to a Hex position.

belonging to PSPACE, as shown above, the problem type of determining whether a given move on a given Hex board is part of a winning strategy is PSPACE-complete.

# Chapter 6

# How to play Hex

This is a text mainly on the history and mathematics of Hex. However, games are there to be played and fortunately the mathematical investigations are useful for the Hex player. In this chapter I will do my best to pass on some advice on the playing of Hex, though my own skills are far from advanced.

It is extremely difficult to give concise and universal directions and what I am able to say here is in serious danger of being obvious or unapplicable because of generality. This difficulty is reflected in that of creating a strong Hex playing program. When I will try just the same, it is because a thesis on Hex would not be complete without it and because there *are* some truly useful tips—mainly some that are extensions of the research approaches described in chapter 8.

When playing Hex, either as a human being or a computer, there are two scopes that must be balanced in order to attain good play. A good player must be able to block an advancing chain and establish his own connection in the close play, but it is equally important to look ahead and prepare traps and escapes. For the computer players there is a very concrete task in deciding the ratio in which to perform game tree analysis and applying more general move evaluations.

Thus, I have divided the chapter into a section on some structural observations and some more general issues to consider.

Naturally, a comprehensive knowledge of the mathematics and the tricks I describe in this chapter is not enough to become a good player. It requires quite a lot of training and perhaps even flair for a player to be able to apply all this advice and ultimately Hex (and many games) is all about responding to the opponent's moves.

Most of the observations in this chapter are inspired by Cameron Browne's *Hex Strategy*[5], random ideas and hints on the internet and not least by real games at the internet game site Little Golem[1].

[1]Little Golem offers Hex on $13 \times 13$ and $19 \times 19$ as well as many other games. Its address is http://littlegolem.net

## 6.1 Recognising structures

Hex, being a tree game, can be played perfectly with a sufficiently large capacity for examining the branches, i.e. the possible lines of play. As has already been elaborated on above, however, it is not possible to perform very deep analysis because of the large branching factor which implies that the workload increases immensely with every deeper level searched.

There are a few structures that are immediately seen to occur again and again in every game. We refer to these as templates because they can be applied all over the board. They consist of a specific position and unambiguous instructions as to responses to the opponent's intrusion into them. Templates establish a 'virtual' connection between two (groups of) possibly occupied cells, one of them perhaps being a side. The point is that the connection can be as good as solid and yet unrealised. Knowing and using these can save a lot of calculations—whether you're a man or a machine—and allow a player to look further ahead.

An important aspect of Hex is to respond to the opponent's moves and to be able to block his attempts at connecting central groups of cells. Blocking is not easily made subject to templates because one has to consider an entire side of the board that the opponent is threatening to connect to. Yet there are a number of similarities and, all other things being equal, we are still able to give advice.

**Bridge moves** The strongest connection between two cells is obviously the completed one. But almost as strong is the connection that is realisable no matter what move the opponent makes. We shall call the simplest version of this virtual connection a bridge because it rests on two cells and spans another pair.

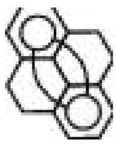


Figure 6.1: A bridge. There are two possible connections between the 'pillars' of the bridge.

It is almost as strong a connection as the one between adjacent cells but spans quite a lot more. This makes the bridge the most fundamental and important local structure to know and recognise.

**Edge templates** When play approaches the edge, a small number of edge templates are extremely efficient. The smallest of these is a version of the bridge in which one end cell is part of the homeland. The templates consist of a small number of cells and a set of instructions on how to respond to the opponent's intrusion into the template.

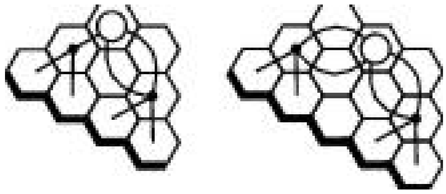


Figure 6.2: Edge templates connecting line three to the edge. Both contain a left and a right part, if the opponent intrudes into one part the player must respond on the pivot point in the other. Notice that the template on the right surrounds an empty cell which may as well be occupied by the opponent.

On the standard board we know of edge templates up to five rows from the edge. Characterising the templates by their distance from the side, the ones furthest away require more space but prove to be equially strong tools. The simplest row three template requires four unoccupied cells along the edge and the row four and five templates require seven and ten unoccupied cells respectively. Obviously there exists edge templates of arbitrary distance to the edge, however, the board may not be big enough to contain them.

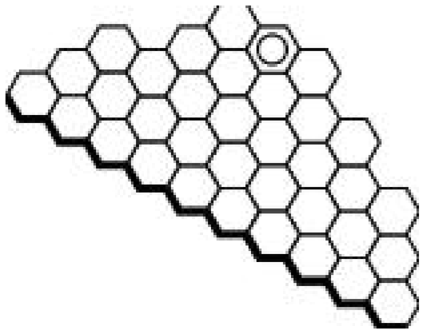


Figure 6.3: The cell position necessary for the minimum row five template. The response instructions are non-trivial and include templates for the first four rows.

All templates can be reflected.

**Ladder** Hein referred to this structure as "rubbing shoulders" because it occurs when a player pushes towards his side and the opponent has no other option but force him along it.[15, 42-12-27] The advancing player can turn the ladder to move in another direction but not force a breakthrough.

When the ladder is moving in the same direction as one of the edges, continued long enough it will connect the other two edges. It can be quite difficult to escape from a ladder once begun unless it meets already occupied

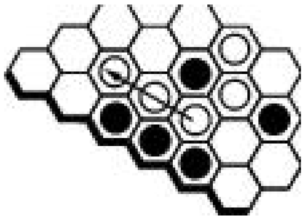


Figure 6.4: The ladder forms along the arrow because the white player cannot reach the edge but keeps pressing towards it. All Black's moves are forced.

cells on its way—and this is the key to its use. These cells are known as ladder escapes and should be played early, threatening to connect another group of cells to the side cf. double threats.

The ladder is in fact a frequently used and quite strong tool. If it is planned well, it is able to force quite a lot of territory into a player's possession.

**Blocking** The hexagonal nature of Hex means that an obstruction cannot simply be placed directly in front of an advancing chain—one can always move around it unless it is being planned a few moves ahead. The classic defense against the bridge move is to block three cells away as shown by the move *2* in figure 6.5 as this gives the blocker two moves before the opponent gets that far.

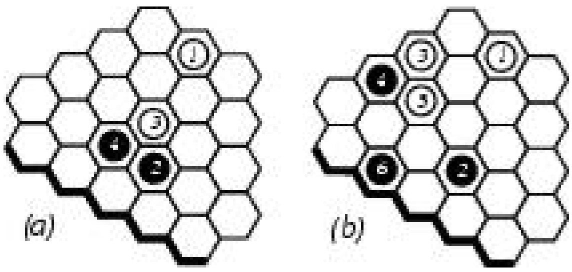



Figure 6.5: Neither the move directly towards the edge (a) nor the move towards the corner (b) manage to pass the classic bridge defense. In the latter case a ladder will form along the second row, moving left.

Hein described a strategy for blocking in *Politiken* as a response to numerous letters that claimed to have found a winning strategy.[15, 43-01-09]

## 6.2 General tips

Using the templates and knowledge from the previous section ensures play at an intermediate level. In order to attain even better play it is highly necessary to plan ahead and notice the traps and openings that the opponent provides.

Obviously this is extremely difficult and relies heavily on intuition and is thus the area that causes Hex programmers great difficulties. Formulating a subtle threat as something which can be discovered by an algorithm is not easy.

The first artificial Hex player was, however, based on general connectivity rather than close, structural play. Claude Shannon's Resistor Network measured electrical current in a net of wires, recommending the player to strengthen saddlepoints. The result was a fairly good Hex player which was easily trapped in the close play.

Modern Hex programs use some degree of tree search and thus have an error free close play but are more easily trapped by good positional play.

This section will feature some general observations that are not easily implemented but good to keep in mind during play.

**Focus on regions** It is impossible to maintain a plan for the entire Hex board at once and thus it can be a good idea to divide it into a few regions, concentrating on one at a time. If a virtual connection is secured from a cell to the edge, there is no reason to consider moves in that region as long as the opponent does not interfere with it.

Also, the importance of each region must be estimated in order to focus attention. One should often respond close to the opponent's last move but occasionally a region is lost and another suddenly becomes the most important.

As the game progresses and territories are won or lost, it is necessary to adapt the territorial divisions and change focus accordingly. It is all about distributing the available computational power most efficiently.

**Claim territory** Along the lines of the previous advice, it is important to claim the unoccupied regions by blocking the opponent's connection to them. Especially during the opening play, it is important to spread out the stones so as to occupy great lands while also limiting the opponent's territories.

Close to the edge it is possible to claim quite a lot of empty cells. Edge templates are good examples of regions claimed. Claiming the acute corners also often proves crucial as ladders are likely to form along edges towards these corners. Stones in these regions will often work as ladder escapes.

**Double threat** Your opponent may also have read this advice and will know how to block your advancing connection. In order to force the connection past a well performed block there must be another connection possible. A good move threatening to form a connection always has an alternative connection in case the opponent should interfere with the original plan.

The bridge is the minimal example of the double threat. In a larger and more efficient scale, a ladder escape should also threaten to connect on its own. What is important is that the two threats do not overlap, as this will make it vulnerable.

**Forcing moves** As seen for instance in the ladder structure, a game of Hex has a lot of moves that are necessary replies because if they are not made, the opponent can win immediately.

Intrusion into a bridge is an obvious instance of a forcing move. The player must reply on the bridge's remaining empty cell to avoid losing the connection. Edge templates also consist of a number of forcing moves.

Forcing moves can be used to a player's advantage, but one must be cautious. When used well a forcing move will be part of a double threat, win important territory or secure the escape of a ladder. A player subject to a forcing move will have two possibilities: To respond to the forcing move and thus allow the opponent to dictate the game or to leave the vulnerable connection open in order to perform a more important move—possibly a move that will force the opponent and thus postpone (or avoid entirely) the destruction of the original connection.

It is often a good idea to desist from forcing a move because it can result in a virtual connection being consolidated. In that case, a player should try to force the opponent to make his virtual connections overlap so as to intrude into two or more of these in a single move.

**Play close** Obviously, it is no use establishing a connection spanning most of the board if the opponent is able to push through a small hole. It is always important to pay attention to where the opponent last played and block his attempts at a connection.

Every move the opponent makes will serve the main purpose of securing a connection and/or weakening yours. In any case, a good move poses an immediate threat that must be responded to.

Of course, in some cases a move has greater consequences in completely different parts of the board than in the immediate proximity. Also, the region that the opponent just played in may not be your most important region.

Cameron Browne concludes his book on Hex strategy with this moral that I will also adopt:

> There is no easy solution to Hex. Between two otherwise equally skilled opponents, the player who is willing to work harder and perform the more thorough lookahead will usually win the game.[5]

# Chapter 7

# Variants

Piet Hein asks in one of his unpublished manuscripts:

> [...]why must the game look just like that? [...] Is it not quite arbitrary…? Could it not just as well…? Why must I adhere to such an arbitrary practice?[sec. B.2.2]

and provides the answer himself: "It is not arbitrary. By its very idea this game could look no other way".[*ibid.*]

However, Hein is not entirely correct in his conclusive statement. He did make a few decisions (however natural) in the invention and in this chapter we shall experiment with a variation of these. Changing the game in different aspects will help us realise what the game is and what it is not. We will modify parameters little by little and examine when the game is no longer Hex, which properties hold and perhaps discover new entirely different, yet related, games.

Hex has only a small number of characteristics that we can modify:

- Mode of movement
- Board layout
- Objective

We can change (or add to) one or more of these—in some cases a change in e.g. board layout implies a new objective etc.

In fact, there is one more aspect to vary, namely the number of players. Three player Hex may for instance be played on a hexagonally shaped board. This game will likely end in deadlock unless saved by the additional rule that when a player can no longer win he is not allowed to make another move.

Games of three or more players will invariably become games of diplomacy and tactics unless much information is hidden. Thus three player Hex is not the game of pure strategy that most two player variants are.

## 7.1 Mode of movement

The mode of movement in Hex refers to the specification that the players alternately place one stone on the board without restrictions. Many additional or different rules are possible but Piet Hein expressed his satisfaction with a game of as little contingency as possible.

Usually this sort of change is introduced in order to keep the fundamental properties while at the same time fixing some imbalance. We saw above that the first player has a winning strategy and it is generally believed that an unconstrained first move is a considerable advantage. Therefore, most changes in mode of movement are concerned with this particular imbalance.

**The swap rule** Once the first player has played his first stone the second player is allowed to swap colours so that he becomes the first player and takes over the opening move. This rule is more often than not used these days because it balances the game excellently and simply in the spirit of the game itself.

The point of this rule is that if the first player plays a strong opening move the second player can just take it over. This forces the first player to play a weak move that the second player is not likely to want. The problem then lies in choosing the best among a number of weak opening moves.

It is clear that this problem forces the first player to open with a move that blurs the strategies of both players and thus adds to the entertainment of the game. Mathematically, the swap rule changes nothing, except that now the second player has a winning strategy, namely to swap if and only if the first player opens with a winning move. The game is still determinable in every move and still one of the players has a possible win that is only lost if he makes a mistake (and the opponent takes advantage of the mistake).

The swap rule is of unknown origin but applies to a majority of games—including that of sharing a pie equally with someone you distrust.

**Agreed starting position** A simple way of starting a game in which it is quite opaque which player has the advantage, is to agree on some initial position of one or more stones. This is an excellent way of handicapping in case of very different levels of players.

**Beck's Hex** Similar to the swap rule, here the second player places the first player's first stone. Anatole Beck showed that opening in one of the acute corners is a losing move thus devising the best forced move.[2]

**Double second move** Another rule of the same sort is to allow the second player to move twice after the opening move. This is a great advantage for the second player who can virtually nullify the opening move.

**Double moves** The double move can be applied to the entire game in order to obtain a quicker game. Double moves allow for more serious threats but equally for easier blocking.

**Kriegspiel Hex** Both players only see their own moves and a referee announces whether a move is possible or not. Being a game of incomplete information, a winning strategy does not exist.

Inspiration for this variant comes from the Chess variant Kriegspiel which was invented in the late nineteenth century by Henry Michael Temple and still enjoys some popularity. The Hex variant was suggested by William McWorter in [17] along with a proof that a winning strategy exists only for a board size $n \leq 3$.

**Limited stone supply** The players have too few stones to fill the board and thus must move stones as the supply runs out. The game may run indefinitely or end in a draw—and will if the players are experienced.

## 7.2 Board

Piet Hein was very excited with his game that had almost invented itself: The hexagons realise the demand that no more than three cells meet anywhere so perfectly and beautifully. But of course there is an infinitude of possible boards that carry the same properties. We will look at just a few of these.

**Isomorphism** The board need not have the hexagonal structure. Indeed, Nash's first version used a quadratic grid with diagonals in one direction added. The board can be scrambled in an infinite number of ways that will render the game unchanged—however, the layout may result in a major difference in the clarity of the game.

$\boldsymbol{m \times n}$ Hex can be generalised to be played on a board of $m \times n$ cells—if $m \neq n$, it is a different game.

It is easy to see that the player connecting the long sides has a considerable advantage and consequently this is a good way to handicap expert players against novices. But how big *is* the advantage? In fact, Martin Gardner devised a pairing strategy for the player with the short connection advantage that will win on the board with $m = n + 1$ even when he moves second. The strategy is for the second player always to move at the cell mirroring the one the first player played. On boards with $m > n + 1$ the second player can simply ignore plays outside the first $n + 1$ rows and play arbitrarily.

**Other tilings** Hex can be played on any board with distinct homelands, some obviously more suitable than others. Cameron Browne describes play on a map of USA on which the players try to connect Canada to Mexico and the Pacific Ocean to the Atlantic respectively. He notes that the player connecting north/south has an obvious advantage and, playing first, will win by opening in California.[5]

Symmetry is no necessity, one could generate a completely random board and have excellent play. As described above, in order to avoid deadlocks, no more than three cells should meet at any point.

**Bridg-It** An early variation of Hex that became a game in its own

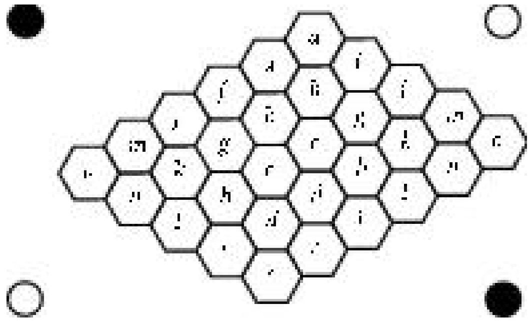


Figure 7.1: Black wins if he plays always on the cell corresponding to White's last move.

right is the game Bridg-It, also known as the Game of Gale for the inventor. The game is played on the edges of two quadratic grids $n \times (n+1)$ and $(n+1) \times n$, mutually displaced half a cell horisontally and vertically. The players play on a grid each trying to establish a connection across his grid on the long direction. The game was solved by Oliver Gross who devised a winning pairing strategy for the first player.[3]

Now, for some altered boards it is no longer possible to keep the objective unchanged. Such is the case with the remaining variants of this section and they are the variants that bear the least resemblance to the original game.

**Unlimited Hex** Hein settled on $11 \times 11$, Nash preferred $14 \times 14$ and modern expert players like to play on $19 \times 19$ or larger for greater challenge. Of course there is no upper limit to the size of the board. An unlimited board is also possible. Since there are now no sides to connect, a new objective must be introduced. A simple suggestion is forming a cycle around an empty or opponent's cell. Five (or any other number) in a row would be another. Ronald Evans described this game under the name Tex.[8]

Unlimited Hex is initially infinite and, depending on the objective, will be likely to end in a draw (if there is an end). For the objectives I suggest, both players can avoid losing and thus force a draw.

**Spherical Hex** It comes naturally to want to play Hex on a sphere. However, there is no mapping of the hexagonal grid on a sphere and so some capers must be made in order to make it a proper game. One possibility is to abandon the hexagonal grid and another is to accept an almost spherical board.

The pattern of pentagons and hexagons normally used for leather footballs known as buckminsterfullerene (which is in fact the molecule $C_{60}$, named after the architect Richard Buckminster Fuller) constitutes a proper mapping but it only has 32 cells and thus makes a very short game. Other patterns are possible which will make the sphere playable—e.g. a randomly generated pattern.

**Modulo Hex** Conserving the hexagonal grid we can pretend to play on a sphere (in fact a torus) by considering opposite sides connected.

For the (almost-)spherical boards we will have to have a new winning condition since a sphere has no edges. We could appoint singular cells that the players must connect but the simplest and most beautiful winning condition is to surround one or more cells (empty or occupied by the opponent) with an unbroken chain. This, however, leaves room for both players winning if play continues after the first surrounding has been established.

**Y** A quite popular game is the game Y that was developed by Craige Schensted and Charles Titus in 1953.[5] It is played on a triangular board of hexagons on which both players attempt to connect all three sides with one chain. It is easy to convince oneself that exactly one player must succeed in this with arguments similar to those used for Hex.

The board of Y can be seen as a subset of a Hex board—and also vice versa—but there are some differences. Y is slightly more pleasing in that the players have exactly the same goal (not just symmetric). But what is more is that the topography of Hex results in considerably different properties of the obtuse and the acute corners. In Y all corners and sides are the same.

## 7.3 Objective

The objective of Hex is difficult to vary unless one accepts completely different games. These three were described by Ronald Evans in [8]. Other more radical variants may combine Hex with Othello, Go or Four-in-a-Row.

**Reverse Hex** One obvious change in objective is to make the one to make a connection the loser. The game is still finite, deterministic and with complete information and so there is a winning strategy. Avoiding a connection seems to require much less planning and trapping and I expect a winning strategy to be possible to find easily. It seems that on boards where $n$ equal first player can win and with $n$ odd second player.

**Vertical Hex** The first player opens along a side and wins if he can connect this stone to the opposite side. This variant must be played on an $n \times m$ board with $m \neq n$. The first player can force at least bridge moves on every move but requires more space than the opponent.

The first player will win exactly by playing the edge templates described in the previous chapter and will thus win on $2\times2$, $3\times4$, $4\times7$ and $5\times10$. This contradicts the conjecture of Evans that the required width of the board is $1 + n \cdot (n-1)/2$.

s **Vex** The first player opens in an obtuse corner and wins if he can connect to one of the opposite sides. The variant is a first player win which follows from Piet Hein's observation that both players cannot be blocked locally.

# Chapter 8

# Recent research

Hex was developed some sixty years ago and it has been shown to be very unlikely that we will ever be able to solve it. Still, quite some efforts are made in the exploration of the game for a number of reasons.

First of all, a problem *seeming* unsolvable has never stopped mathematicians from trying anyway—and trying for hundreds of years.

Secondly, recent efforts have largely been focusing on *approaching* a solution. Approximated solutions have been achieved by artificial Hex players that are now able to offer good play. Also the OHex database is able to give some (however few) hints as to where to look for a solution to Hex.

Thirdly, it is usually so that results in one area yields discoveries that are applicable to other areas and this is probably one of the main reasons for the continued digging for the needle in the haystack. Ultimately, Hex is just the concrete occasion of a more general research of algorithmic approaches to game trees with large branching.

Finally, the mathematics of Hex is a good excuse for playing games during working hours.

This chapter will take a tour around the main areas of research being carried out these years, demonstrating that Hex is a game worth exploring more than sixty years after its invention.

## 8.1 The OHex database

Kevin O'Gorman at California Polytechnic State University has commenced the Sisyphean project of drawing up entire Hex game trees. He is building a database, named OHex, of played games to be used to evaluate a given board position and suggest the best next move. Each game is thus represented by a path from the root of the game tree in question to a leaf.

The website at http://hex.kosmanor.com/hex-bin/board features an interactive Hex board. Each cell is labelled as a possible win or a possible loss.

It is possible there to play through all the games known to the database, for each move OHex giving an estimate of the next best move.

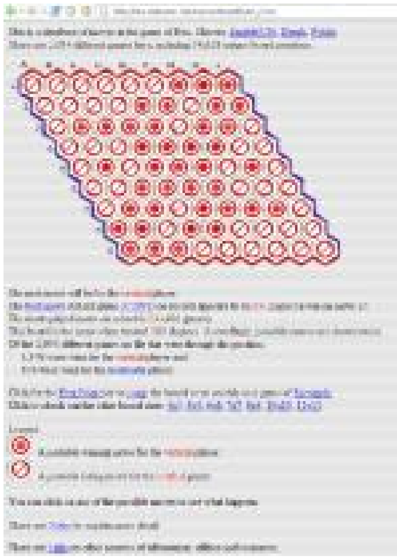


Figure 8.1: Screenshot from the OHex database at http://hex.kosmanor.com/hex-bin/board. The board shows estimates of whether the different cells are winning or losing moves.

O'Gorman records games on the boards $4 \times 4$ up to $11 \times 11$ and thus has a tremendous number of possible board positions to store. His own optimistic estimate for the number of relevant board positions on the $10 \times 10$ alone is $10^{39}$. Add the other boards and he will need storage the size of a city. It is not necessary to store all possible distributions of stones since a major part of all games end with less than half the cells occupied—and a 180 degrees rotation of the board renders it unchanged thus reducing the necessary tree by one half.

Nevertheless, O'Gorman's project is of some interest. At the very least, it can be an aid when playing Hex. The stored games are taken mostly from actually played games at some of the public online websites that offer Hex. Only a few are generated automatically by Hex playing computers. This means that the database can in fact be of help when it contains a major part of experienced players' responses to good moves.

For obvious reasons, the branches close to the root have been better examined than the ones far away. It is not very likely that a particular game has been played often before with the vast number of possible games. This is evident when trying to use OHex to aid in a real game. OHex may thus be of most help in the first few opening moves.

At the moment the database contains less than 30 million board positions which is an infinitesimal fraction of the number of possible positions. The table below lists the current numbers for the different board sizes. The greater number of sizes 7 and 10 games are due to the accessibity of different implementations of Hex online.

| size | games | positions |
|---|---|---|
| 4 | 365 | 1661 |
| 5 | 164 | 1298 |
| 6 | 288 | 3417 |
| 7 | 1963 | 33954 |
| 8 | 137 | 3268 |
| 9 | 2054 | 54628 |
| 10 | 798395 | 24507299 |
| 11 | 109025 | 3972692 |

Until the OHex database contains a significant part of the possible games its main use is to estimate the quality of different moves by comparison to what others have chosen. A well examined move, corresponding to one often occuring in the database, that OHex expects to win, will very likely be a very good move since many players have thought so.

Let us for a moment imagine that Kevin O'Gorman manages to store the complete game trees for many sizes of boards; the general solution to Hex will still be inaccessible since the database cannot generalise its strategies in any way but only formulate them in terms of full game trees.

I contributed (slightly) to the project by translating the website into Danish.

## 8.2 The Decomposition Method

Another attempt at producing perfect solutions is being made by Jing Yang who is a computer scientist at The University of Manitoba, Canada. He has succeeded in tracking a winning strategy for a number of specific openings on different boards, most notably one on the $9 \times 9$ board. We already know from Nash's proof that there *are* winning strategies for the first player on any board and Yang supplies us with some of these.

The approach taken by Yang is a method which consists in decomposing a board position into disjoint groups of cells each constituting a secure connection and these in conjunction a winning connection. The method is closely related to the development of edge templates but Yang's templates also consider the inner board and allow the opponent's pieces when necessary. The decomposition method also resembles what most players do unconsciously when playing. Experienced players recognise structures and simply remember local winning strategies and thus save a number of computations.

Yang has applied his method so far to boards of up to $9 \times 9$ manually. The approach is to consider a specific opening and then decompose the board into a number of subgames, each containing specific countermoves to any move by the opponent. In turn, these subgames will decompose into subgames of their own etc.

The actual construction of the patterns for the complete decomposition is a difficult process that includes considering any move the opponent might make. Yang describes taking advantage of the 'sudden death' property of Hex that implies that many positions have moves that are more or less forced thus eliminating large branches of the game tree.[22]

### 8.2.1 Decomposition of $5 \times 5$ Hex

As an example of the decomposition method, I have made a reconstruction of a winning strategy for the opening D3 on the $5 \times 5$ board. Accordingly, this is only $\frac{1}{25}$ of the complete solution for this board.

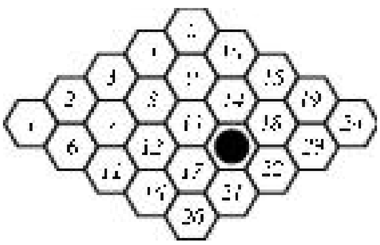
The pseudocode is Yang's and self-explanatory. The algorithm stores a number of local patterns in a list called `SumOfLocalGames`; `WhiteMove` holds the name of a white intrusion into one of these and `BlackMove` is the output i.e. black player's response. For each move by black, the active local pattern is removed from `SumOfLocalGames` and one or more new smaller ones are added.

Since this method is a sure-fire winning system, any response can be given to a white move outside the `SumOfLocalGames`. No attempts have been made to make this strategy win quickly. What is important is that it is certain to win.

Black begins and connects upper right to lower left.

---

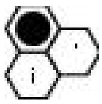
**LocalPattern1**



```
If(WhiteMove == 18||19||22||23||24) {
 BlackMove = 14;
 SumOfLocalGames = SumOfLocalGames - LocalPattern1
  + LocalPattern2(10,15)
  + LocalPattern5(1,2,3,4,5,6,7,8,9,11,12,13,16,17,20,21); }
```

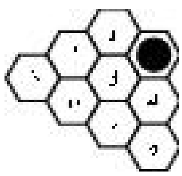
```
else if(WhiteMove == 10||14||15) {
 BlackMove = 23;
 SumOfLocalGames = SumOfLocalGames - LocalPattern1
  + LocalPattern2(18,22) + LocalPattern2(19,24)
  + LocalPattern5(1,2,3,4,5,6,7,8,9,11,12,13,16,17,20,21); }
else if(WhiteMove == 17||20||21) {
 BlackMove = 8;
 SumOfLocalGames = SumOfLocalGames - LocalPattern1
  + LocalPattern3(22,23,18,14,24,19,15,10)
  + LocalPattern3(3,2,7,12,1,6,11,16)
  + LocalPattern4(4,13,9,5); }
else {
 BlackMove = 20;
 SumOfLocalGames = SumOfLocalGames - LocalPattern1
  + LocalPattern2(17,21)
  + LocalPattern3(22,24,18,14,24,19,15,10); }
```

**LocalPattern2**



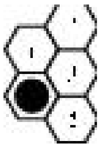
```
If(WhiteMove == 1) {
 BlackMove = 2;
 SumOfLocalGames = SumOfLocalGames - LocalPattern2; }
else if(WhiteMove == 2) {
 BlackMove = 1;
 SumOfLocalGames = SumOfLocalGames - LocalPattern2; }
```

**LocalPattern3**



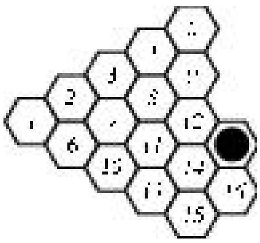
```
If(WhiteMove == 1||3||4||7||8) {
 BlackMove = 2;
 SumOfLocalGames = SumOfLocalGames - LocalPattern3
  + LocalPattern2(5,6); }
else if(WhiteMove == 2||5||6) {
 BlackMove = 4;
 SumOfLocalGames = SumOfLocalGames - LocalPattern3
  + LocalPattern2(1,3) + LocalPattern2(7,8); }
```

**LocalPattern4**



```
If(WhiteMove == 4) {
 BlackMove = 2;
 SumOfLocalGames = SumOfLocalGames - LocalPattern4
  + LocalPattern2(1,3); }
else if(WhiteMove == 1||2||3) {
 BlackMove = 4;
 SumOfLocalGames = SumOfLocalGames - LocalPattern4; }
```

**LocalPattern5**



```
If(WhiteMove == 14||15||16) {
 BlackMove = 8;
 SumOfLocalGames = SumOfLocalGames - LocalPattern5
  + LocalPattern3(3,2,7,11,1,6,10,13)
  + LocalPattern4(4,5,9,12); }
else {
 BlackMove = 15;
 SumOfLocalGames = SumOfLocalGames - LocalPattern5
  + LocalPattern2(14,16); }
```

---

Thanks to the modest size of the strategy one easily follows and validates it. My $5 \times 5$ solution relies on only 5 local patterns, whereas the one solution that Yang has developed for $9 \times 9$ requires 715 local patterns, suggesting that this method, although much more efficient than minimax search, also undergoes exponential growth.

It is likely that once one has a reasonable supply of known local patterns it becomes relatively easy to run through them looking for something to fit into a board one is decomposing or even playing on. This library of local patterns implies a one-level search as opposed to ordinary tree searches.

Applied to end-game these results are potentially large improvements to artificial Hex players. This has not been examined so far though.

Yang expects to be able to automate the process of developing the local patterns which is necessary if his contribution is to have any significance, as the work required is quite large. Despite the fact that the patterns already developed obviously constitute quite a lot of those necessary for the bigger boards much creativity and intuition is required in the development—if we are not to reduce the task to the regular brute force search.

## 8.3 Hex playing computers

Whereas O'Gorman and Yang work on the mapping of parts of perfect solutions, others have taken a heuristic approach in their research.

As early as 1953 Claude E. Shannon and E. F. Moore developed the world's first Hex playing device named the Shannon Resistor Network.[11] The idea is some fluid or current—or some virtual flow—flowing in a network between two sides. Each vertex in the network has unit resistance or capacity and the players enlarge or reduce this resistance or capacity respectively. Determining the current across the network will give a hint to strategic vertices.

The result was a Hex player that, according to Shannon, showed good positional judgement but was weak in end-game combinatorial play.

The idea of viewing the game board as a potential field does in fact offer a general idea about the value of the current position. However, as was Shannon's observation, the network model cannot predict traps or plan forcing moves as expert human players or even advanced computer programs can.

### Algorithmic search

With the development of the computer Hex was easily implemented but proved quite difficult to play automatically. The website of the International Computer Games Association lists a small number of Hex playing programs of which most either have no artificial player or are implementations of a known winning strategy on a small board just like my own example of the decomposition method.

Even with the early and great efforts to create Chess computers no significant progress was seen for decades because the objective was primarily for larger and faster computers and not for better algorithms. We have seen above that Hex has a game tree very different from Chess in that it is at most $n^2$ levels deep but in turn branches heavily during the first part of the game. This results in a massive demand for extra capacity for just one extra level in a tree search.

The basic game tree search is known as *minimax search*. Each node will be assigned a value recursively, the maximum value of its children if it is first player choice and minimum otherwise—given explicitly in chapter 4.

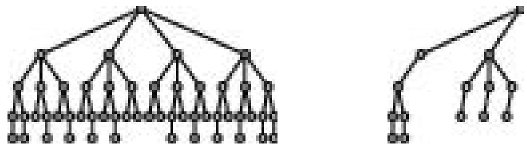


Figure 8.2: Left: The game tree of a $2\times 2$ Hex game in an arbitrary ordering. The minimax algorithm must visit all 52 nodes before determining the value of the game and the best move. Right: By this ordering of moves, the alpha-beta-algorithm succeeds in truncating the game tree of $2 \times 2$ Hex to 13 nodes in determining optimum result for both players—not when devising a winning strategy.

*Alpha-beta search* is similar but usually somewhat more efficient in that it will start comparing values at the parent level too and cut off a subtree if no better result than that already obtained is possible. Alpha-beta search may reduce the time consumption but only if the values in the game tree occur in an order that allows it. However, even with all the best choices on all the maximising player's nodes, all the minimising player's nodes must be visited for an explicit winning strategy—maintaining the need for exponential time.

Both of the tree search algorithms fail on their own because they work recursively and thus must visit the leaves of the tree in order to even start assigning values to other nodes in the tree. They determine the values from the bottom, so to speak. In the case of Hex, with a wide and shallow tree, both minimax and alpha-beta-search in themselves have provided no practical results.

It has been realised that man will not be beaten simply by brute calculating force. Eventually weighted searches were introduced so that interesting branches could be examined deeper on the expense of less interesting ones. Thus, a difficult quest for evalution methods arose searching for a way to estimate values and to determine interesting moves.

**Selective search**

The next real step towards a good Hex player was taken by Jack van Rijswijck at The University of Alberta. Rijswijck developed the Hex playing program called Queenbee as a part of a Master thesis.[20] The fundamental idea was a selective search guided by an evaluation function based on graph distance.

According to its own website (http://www.cs.ualberta.ca/~queenbee/), Queenbee was first built in 1994 but some of the core functions have changed and improved significantly over the years. The particular evaluation function that made Queenbee notable is called *two-distance*. Rijswijck describes it as "best second-best alternative".

Basically, Queenbee discovers possible connections and determines the minimum number of moves necessary to complete them. It can be assumed that the opponent will always attack the best connection, namely the one requiring fewest moves for completion and therefore the second-best becomes a good choice for describing the strength of a board position. The distance from one cell to another or to the side is defined recursively as one more than the 'smallest but one' of the neighbour cells' distances.

Queenbee plays at the level of intermediate human players and was exceeded by a stronger computer program named Hexy, developed by Vadim V. Anshelevich in 1999. Hexy is based on deducing virtual connections in subgames, an algorithm known as hierarchical search—in short H-search.

### H-search

The key idea behind H-search is to apply two rules alternately to discover connections that are secure but still need to be realised. The well-known bridge is one of the most simple virtual connections.

A set of cells contains a *virtual connection* between two distinct cells if and only if the connection can be realised, even if the opponent moves first. A *virtual semi-connection* requires the player to move first. The rules described as the OR rule and the AND rule combine virtual connections and virtual semi-connections thus heavily reducing the tree search necessary for good play.[1]

**The AND rule** Two virtual connections with exactly one end cell in common are reducible to one virtual connection if the common cell is occupied; a virtual semi-connection if the common cell is empty.

**The OR rule** Two virtual semi-connections with only their two ends in common are equal to a virtual connection.

The bridge is an excellent example of the OR rule applied to two minimal virtual semi-connections whereas the AND rule can be demonstrated by the example of fig 8.3.

This figure does in fact show a simple example of hierarchical search (short: H-search). Initially we have a set of cells in which the only connections are between neighbours. Applying the AND rule gives us the four virtual semi-connections; the OR rule used twice establishes two virtual connections. Finally, the AND rule once more shows that the two ends are in fact virtually connected via this repeated use of two simple deduction rules.

H-search may continue until no more virtual connections or semi-connections are deducible or until a winning virtual connection is established. In the implementation of H-search in Hexy, a limit of 20 iterations of

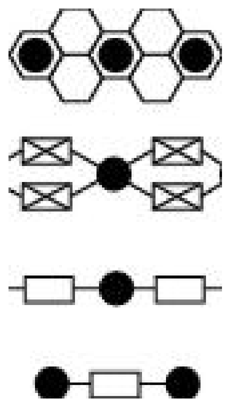


Figure 8.3: The OR and the AND rule demonstrated: The board position is translated into four virtual semi-connections which are reduced to two virtual connections in turn equal to just one strong virtual connection.

the deduction rules have been set up in order for it to execute in reasonably short time.

Anshelevich himself showed that there are virtual connections that the AND and the OR rules are unable to discover.[1] However, Rune Rasmussen and Frederic Maire of Queensland University of Technology have showed how to extend the search beyond some failed OR connections.[18]

The extension of the H-search takes advantage of something similar to what Jing Yang exploits, namely the forced move regions. In the case of H-search, the OR rule fails if the virtual semi-connections overlap and the opponent must play in this region for the connection to fail. Only considering these *must-play* regions, Rasmussen and Maire have devised an improvement to the algorithm which reduces the interesting branches to only those that overlap.

H-search has also been implemented in Six by Gábor Melis with a resulting Hex player that is currently the strongest available.

## 8.4 Expected results

On the basis of the chapters above, we can safely say Hex will not be solved one of these days. So let us conjecture a little about what we *can* expect to see in the future regarding Hex.

First of all, Six probably will not be able to keep the status as the strongest artificial Hex player. Six has been released as open source so it is likely that someone will modify the algorithms slightly for improved performance. We might also see a completely new program, combining some of the techniques described above; especially Jing Yang's decomposition patterns have potential for one such, but the knowledge collected about opening strengths by both OHex and Queenbee could provide details about the strategic values of the Hex board. A detailed opening library may well be developed soon.

We can also expect Jing Yang or others to accomplish larger decompositions still. If someone manages to automate the decomposition process this method will surely provide extremely strong information for the programmers of artificial Hex players.

OHex can be expected to grow and perhaps even faster with the success of online gaming websites and maybe also some automation. That it will probably not gain us any deep insights have been elaborated upon above.

The large group of problems in the complexity class PSPACE will probably grow and attention to it remain extensive. General belief is that no real progress is possible in this area but there may be some gains in examining and possibly translating the heuristics of other PSPACE-hard problems.

Should the unlikely event occur that someone works out a general strategy that seems to work against all the best Hex players, there is still not enough time available for it to be anything near properly verified. We can relatively easily expose it to very diverse problems in PSPACE, though. If it holds against all these, however unlikely, we have come a long way.

# Chapter 9

# Conclusion

We have taken a tour from history, over game theory, graph theory and complexity theory to game analysis and computer science. I have tried to gather all aspects of Hex into one presentation in a balance between shallowness and meticulousness.

I hope to have succeeded in making a significant contribution to the scarce literature on Hex. My discoveries may be only a small part of the full picture but I hope to have provided future researchers with a basis that clearly points out the areas worth looking into—and with hints as to where to look for the information that I did not provide.

I have stated that Hex will probably not be solved in the near or distant future, but I expect the game to be the subject of much research anyway. Game theory is still a quite young discipline and many problems have not been examined extensively with an algorithmic approach yet. We may well see the results from Hex extended and generalised into highly different realms over the next years.

Hex has proven itself a research area in its own right with deep and far-reaching perspectives. But even if you don't care about perspectives, Hex is a genuinely entertaining game that anyone can play.

# Appendix A

# Hex to a wider audience

One part of this thesis has been to promote Hex to a wider audience outside the limited circles of mathematicians. I have used three means for this promotion.

Quite early in the writing process of this thesis I established a website that has developed along with my research and writing.

About halfway through the thesis I wrote a feature article for *Politiken.* They did not accept it and I rewrote it for a competition.

Finally, I and a fellow student have begun writing a textbook on Hex for the upper secondary school.

Below is a print of the website, a copy of the feature article and a preliminary synopsis for the textbook.

## A.1 Website

As one of my first actions when starting the work on this thesis I published a website on which I have been posting excerpts from my work in order to get comments, ideas and just to get in touch with other people with an interest in Hex. The address is

http://maarup.net/thomas/hex/

The website has more than a hundred views per week thanks to the numerous ingoing links and good search results. I have been contacted a few times by people with questions and comments.

The website is targeting people who already know Hex and are familiar with some of its mathematical properties or at least with some mathematical terminology.

http://maarup.net/thomas/hex/

# Hex

Home

Cannot end in a draw
First player wins
Complexity
PSPACE-complete
The size of Hex
Classifying Hex

The invention of Hex
Hex's first appearance
Second appearance
Piet Hein
John F. Nash

Decomposition method

Links
Terminology
Bibliography
About this site

**Rules**
Black and white take turns occupying any unoccupied cell. The one to make an unbroken connection between his two sides wins. Board can be of any size, standard is 11x11.

**Download board**
7x7: [pdf]
11x11: [pdf], [eps]

**Last modified**
Wed, 18 May 2005 09:05:38 GMT

**Log**
May 14, 05: Hein added
May 12, 05: Nash biography and First player wins updated
Apr. 6, 05: Nash added
Mar. 8, 05: PSPACE-complete added
Feb. 15, 05: Decomposition added

## Status

This site is being built around a thesis on Hex, an abstract board game with simple rules but which is hard to master. I will be posting parts of my studies and research here over the next months. I'm listing some of the troubles that I run into and issues that I'm working on - please let me know if you have any info to offer.

- Right now my attention is directed at the recent work on Hex: Six, Hexy, Queenbee, OHex and the partial solutions obtained by Jing Yang. Are there other efforts that I forgot?
- I don't know why the series of articles in *Politiken* stopped in August 1943 but I assume that everybody just got bored with it.
- Politiken refused my feature article, a popular presentation of Hex, due to too much material. Download it (in Danish) if you like. I'll review it and try again this spring.

## Hex cannot end in a draw

One of the reasons that Hex is a very satisfying game is the property that it cannot end in a draw. Piet Hein stated in his first presentation of the game this property but as an entirely obvious implication of the hexagonal board. The following proves what Hein saw intuitively.

This proof resembles the proof given by David Gale in his article on Hex and the Brouwer Fixed-Point Theorem. It is based on well-known facts of graph theory. A drawing to illustrate some points is coming up.

Let G be a planar 2-connected graph in which all vertices have degree 2 or 3. Each area within G corresponds to a cell on a Hex board. We can assume that all cells are occupied because a game of Hex cannot end before a player wins or no cell is available. Now make a subgraph G' of G created by colouring the edges that separate two areas of different colour on the Hex board.

This G' will consist of vertices of value 0 or 2 because of the fact that no vertex can be visited twice. Exactly four vertices will have a value of 1 namely the corners: a, b, c and d. It is a fact that graphs with vertices of value no more than 2 will consist of isolated vertices, simple cycles and simple paths (where simple means that they do not split). Since G' has exactly 4 vertices of value 1 there must be exactly two of these paths.

It follows easily that the two paths must connect a-b and c-d or a-d and b-c. In the first case white will have won, in the second black.

## First player wins

Assuming that there is a winning strategy for one of the players in Hex it is quite easy to show that this must apply to the first player. Unfortunately this is a proof of existence and does not give us any hint about the actual

Figure A.1: A screen dump from my website on Hex. The full and updated text is found at http://maarup.net/thomas/hex.

## A.2 Popular article

The following is a feature article in Danish that I wrote for *Politiken.* They did not accept it so I rewrote it to participate in a competition on scientific communication at the Faculty of Science and Technology, University of Southern Denmark.

The objective of the competition was to write a popular article about a scientific subject so that it might fit into a newspaper. The article recieved honourable mention.

The intended audience is people without significant knowledge of mathematics or game theory and a compromise between the scientific strictness and a popular tone was necessary. The article is only available in Danish.

---

"Pludselig midt i morgendæmringen vågnede et spil og krævede at blive født. I dag er det modent til at slippes ud i verden, og det er det, som her i al julelig uskyldighed skal forsøges."

*

Sådan begynder Piet Hein en introduktion til et spil, som han opfandt i julen 1942 og præsenterede for Politikens læsere i mere end halvtreds rubrikker igennem et halvt år fra 26. december 1942. Spillet er så enkelt, at man kan undre sig over, at det ikke er en af de tusind år gamle klassikere som skak og backgammon.

Trods enkelheden er spillet genstand for moderne forskning i matematik og datalogi—og kan måske give ny viden om kunstig intelligens og løse en lang række optimeringsproblemer indenfor både industri, logistik og planlægning.

Samtidig er det et dybt underholdende spil som kan spilles uden forudsætninger.

*

Hex er et spil for to spillere, og det spilles på en plade af sekskanter—som en tavle bivoks—med form som en skæv firkant med fire lige store sider, en rombe.

Den ene spiller ejer to modstående sider, og den anden spiller ejer de to andre. Man skiftes til at placere en brik på et valgfrit felt, og den, der med en ubrudt kæde af brikker skaber kontakt mellem sine to sider vinder spillet. Reglerne er altså enormt simple, og alligevel viser spillet sig at være på niveau med skak i dybde og kompleksitet. Piet Hein valgte oprindeligt at spille på en rombe med 11 felter langs hver side, men man kan også med fordel spille på større brætter, når man er øvet.

*

Hex samler på forunderlig vis en hel række spændende historiske, matematiske og bare underholdende egenskaber, som det er typisk for den alsidige Piet Hein. Spillet hed 'Polygon', da Piet Hein første gang præsenterede det for et publikum.

Han havde i en længere periode arbejdet med spil og forsøgt at finde frem til essensen af gode spil og resultatet var Hex.

Igennem seks måneder skrev Piet Hein mere end halvtreds stykker om Hex i Politiken og udskrev desuden præmiekonkurrencer om 50 og 100 kr. for de bedste løsninger på fiktive spilsituationer.

*

På Princeton University i USA udviklede John Nash (kendt fra filmen *A Beautiful Mind*) det samme spil i 1949—bare syv år efter Piet Hein. Nash modtog i 1994 Nobelprisen i økonomi for sit bidrag til økonomisk spilteori. På Princeton hed spillet 'Nash', eller ifølge en populær, men tvivlsom, morsomhed 'John', fordi man spillede på kollegiebadeværelsernes sekskantede fliser—John er amerikansk slang for lokum. Det var et amerikansk spilfirma, der gav spillet navnet Hex og det er det navn, det kendes under i dag.

Da Piet Hein senere fik spillet produceret i ædeltræ, et flot eksempel på Danish Design, kaldte han det med vanlig opfindsomhed Con-Tac-Tix, som et ordspil dels på spillets to hovedelementer—kontakt og taktik—og dels på tick-tack-toe, som er det engelske navn for kryds og bolle.

*

Piet Hein fortæller i Politiken, at spillet bygger på to simple ideer, og "da disse havde fundet hinanden, var ikke alene ideen født men hele spillet udformet". Den ene af de to ideer bag Hex er en gammel matematisk sætning kendt som firefarvesætningen.

Sætningen siger, at det er muligt at farve ethvert landkort med kun fire farver, således at to nabolande aldrig har samme farve. Det har drillet matematikere i flere hundrede år og blev først bevist i 1976 som det første større problem, der blev løst ved hjælp af en computer.

Det er en konsekvens af firefarvesætningen at det ikke kan lykkes for begge spillere at lave en forbindelse henover spillebrættet, det vil sige at spillet ikke kan ende med to vindere.

Piet Heins anden ide var den, som sikrer at heller ikke begge taber. Mindst én af spillerne vil få etableret en forbindelse. Det er en egenskab ved sekskantstrukturen. Havde brættet for eksempel været kvadreret ville de fleste spil ende uafgjort—skakbrættet kan ses som et eksempel på et uafgjort spil mellem sort og hvid.

Piet Hein fortæller, at med målet “at forbinde to modstående sider” og “brættet med sekskanter” har spillet nærmest opfundet sig selv. Så er reglerne næsten skrevet. Hans eneste bidrag har været at bede spillerne skiftes til at placere en brik.

Da det er sådan at netop én spiller opnår forbindelse på tværs, følger det at offensivt spil og defensivt spil kan være lige godt. Lykkes det at forhindre modspilleren i at skabe forbindelse, vil man opdage, at man selv, måske uden at vide det, har fået sin egen forbindelse lavet. Kan man ikke gennemskue, hvordan man skal opnå sin egen forbindelse, kan man altså blot nøjes med at koncentrere sig om at stikke en kæp i hjulet på modstanderen.

*

John Nash viste, at der, for den som starter, findes en strategi, som garanterer sejr, uanset hvor god modspilleren er. Det eneste problem er, at ingen endnu har fundet denne optimale strategi. Trods beviset for at der findes en vindende strategi i Hex, er det ikke sandsynligt, at den nogensinde vil blive opdaget.

Forsøger man at finde den, er den oplagte opskrift at spille alle de mulige spil igennem—der er jo højst så mange mulige træk som der er felter på brættet. Det er ikke nødvendigvis den nemmeste måde at gøre det på—men den bedste som vi kender. Fremgangsmåden viser sig dog hurtigt at give alvorlige problemer. Regner man ud hvor mange træk, man skal igennem bare på standardbrættet vil antallet i disse spalter fylde adskillige linjer, det er et tal med næsten 200 cifre. Hvis man sætter en rigtig hurtig computer til at undersøge en million spil i sekundet vil solen være udbrændt og universet kollapset, længe inden den er blevet færdig. Og det var kun standardbrættet—herefter kommer brætter i alle andre mulige størrelser, før man har en fuldkommen løsning.

Det ser meget ud til, at der slet ikke findes en nem måde at finde en vinderstrategi på. Hex tilhører nemlig en type af problemer i matematikken, som er særligt genstridige. Problemerne har det til fælles, at der ikke kendes nemme løsninger. Desuden er det sådan, at hvis man skulle finde en løsning på blot ét af problemerne, kan den løsning oversættes til alle de andre problemer. Sådan en løsning er der udlovet en dusør på en million dollars for.

*

Der er naturligvis matematikere og dataloger, der faktisk leder efter videnskabelige metoder til at blive bedre Hex-spillere. For eksempel ved at få computere til at spille Hex, ligesom IBM-computeren Deep Blue, der præsterede at slå stormesteren Garry Kasparov i skak i 1997.

Man kan forsøge at lære en computer at spille Hex. For eksempel ved at studere strukturer, der går igen, så computeren kan lære en sekvens af træk udenad. En anden mulighed er at beregne, hvor på brættet der er den tilsyneladende bedste forbindelse og så bygge videre på den. Der er forskellige tilgange, og i de årlige computermesterskaber i Hex bliver de forsøgt mod hinanden. Sidste år var det et ungarsk computerprogram, der løb med sejren. Men det er endnu langt fra at kunne besejre de bedste menneskelige spillere, fordi menneskets hjerne arbejder på et langt mere abstrakt niveau, end det kan lykkes at programmere en computer.

*

Matematikerne fortsætter ufortrødent det vanskelige arbejde, fordi det ofte er sådan, at store gennembrud sker, når man arbejder med noget uden helt at kende perspektiverne.

Sikkert er det, at arbejdet med Hex giver et forskningsmæssigt afkast i form af stærkere søgealgoritmer og resultater, der kan bruges i forskning i kunstig intelligens. En fuldstændig løsning af Hex—det vil sige en kortlægning af alle vinderstrategier—vil være et stort bidrag til utallige problemer og være et gennembrud i optimeringsteori.

For almindelige mennesker gælder først og fremmest, at Hex er et godt og tilfredsstillende spil. Men det er også ofte sådan, at spil er en camoufleret måde at træne matematik på. Tit er det elementær hovedregning og hukommelse, men generelt styrker mange spil abstraktionsevnen. I Hex benytter man specifikt mønstergenkendelse og det at kunne se nogle træk frem, som begge er væsentlige matematiske analyseredskaber.

*

Piet Hein døde i 1996, i år ville han være fyldt 100 år og det er en anledning til at mindes og beundre hans kreativitet og alsidighed. Grukkene og Superellipsen er blevet Piet Heins kendteste varemærker, men så absolut ikke hans eneste succeser. Hex er blot endnu et eksempel på hans evne til at lave de mest simple konstruktioner, som bygger bro mellem det let tilgængelige og videnskabens dybe perspektiver. Så er man ligeglad med perspektiver, kan man bare nyde Hex som et dybt underholdende spil der kan spilles af enhver.

*

Thomas Maarup skriver speciale om Hex, dets historie, matematiske egenskaber og konsekvenser. Læs mere på maarup.net/thomas/hex.

## A.3 Teaching material

I have commenced work on a textbook about Hex aimed at the upper secondary school. Co-author is Steffen M. Iversen, a fellow student who is mainly interested in the mathematics education. There is a clear division of work between us as I provide the contents and he has the educational, didactical focus on structure, methods etc.

The book's ambition is to be an entertaining introduction to graph theory and some game theory but also to show the potential of mathematics as a subject of good stories. The result so far is a synopsis and draughts for the first two chapters.

[A synopsis in Danish was included in the original paper.]

## Appendix B

# Additional material

## B.1 Polygon in Politiken

Hex was first introduced in the Danish newspaper *Politiken* under the name Polygon. This is a complete list of what the column presented.

**1942**

**December 26** Introduction to the principles of connection and the hexagonal pattern. Rules description and strategic hints. Example opening and problem 1 which is a prize contest.

**December 27** New opening example. Analysis of boards $1 \times 1$ through $5 \times 5$. Description of "rubbing shoulders" (ladder). Problem 2.

**December 28** New opening. Problem 3.

**December 29** New (amusing) opening. Problems 4 and 5.

**December 30** New opening. Problem 6.

**December 31** New opening. Problem 7.

**1943**

**January 1** Ending of prize contest and introduction to a new prize contest about playing a good game. *Politiken*'s playing pads announced.

**January 2** Problem 8.

**January 3** Problem 9.

**January 4** Problem 10.

**January 5** Problem 11.

**January 6** Problem 12.

**January 7** Problem 13.

**January 8** Problem 14.

**January 9** How to block an advancing row. Problem 15.

**January 10** Counter-offensive to the bridge. Problem 16.

**January 11** Problem 17.

**January 12** Advertisement: Train compartment.

**January 13** Introduction of cell labelling. Problem 18.

**January 14** Problem 19.

**January 15** Problem 20.

**January 16** Advertisement: Strategy?

**January 17** Advertisement: All-Clear.

**January 20** Walk-through of the prize winner.

**January 23** Problem 21.

**January 26** Advertisement: At the dentist's.

**January 27** The bridge move reintroduced, also as a countermove. Problem 22.

**January 30** Invitation to a demonstration at *Politiken* February 1. Problem 23.

**February 3** Problem 24.

**February 6** Problem 25.

**February 10** Problem 26.

**February 13** Problem 27.

**February 17** Problem 28.

**February 20** Problem 29.

**February 25** Problem 30.

**February 27** Problem 31.

**March 3** Problem 32.

**March 6** Announcement of tomorrow's prize contest.

**March 7** Prize contest about best played game with a given opening.

**March 10** Problem 33.

**March 13** Solution to problem 33.

**March 17** Problem 34.

**March 20** Problem 35.

**March 21** Advertisement: Cinema.

**March 24** Problem 36.

**March 27** Announcement of prize winner.

**March 31** Problem 37.

**April 3** Problem 38.

**April 7** Problem 39.

**April 10** Problem 40.

**April 14** Problem 41.

**May 12** Problem 43.

**June 16** Problem 44.

**June 23** Problem 45.

**June 30** Problem 46.

**July 9** Problem 47.

**July 21** Problem 48.

**August 11** Problem 49.

## B.2 Hein's manuscripts

I have discovered some previously unpublished manuscripts by Piet Hein. Some of these were quite interesting and reveal some of Hein's thoughts on Hex. This section contains a translation of three hand-written manuscripts:

**The Parenthesis** I believe that this is Hein's lecture at the association at The University of Copenhagen where Hex was introduced to a public for the very first time.

**Demonstration** This is the manuscript for a demonstration of Hex. *Politiken* announced such an event on February 1, 1943 and this is probably it.

**Play with different people** This short paper may be a sketch for the column in *Politiken* or just a realisation committed to paper. It describes the difficulties in finding the winning strategy.

The manuscripts are all without titles and so these are my additions. A hand-written manuscript is not easily printed and so educated guesses were necessary with a few of Hein's symbols and squiggles. Also, his underlining has been replaced with *italics*. My comments appear in square brackets.

### B.2.1 Lecture at The Parenthesis

*a very small contribution to clearing out in the miserable conditions with regards to mathematics and physics as expressed by Aage Bohr.* [Added in the top margin]

What I intend to provide tonight is merely a sketch of a thought to an introduction to a game. The idea is to look at mathematics as a game, and the game is a simple example of looking at games as mathematics. I am not sure how much spiritual nourishment there is in it for you, so it would put me at ease if you would continue eating and drinking.

A couple of years ago a literary critic of the kind who—rightly so—sees his own elevated position in deriding the human ability called intelligence wrote in an article on something completely different that mathematics cannot offer us anything than what was already in the premises. This is quite correct. And it sheds a light of foolishness on the business of mathematics. And in this article he did go on as if mathematics with this remark once and for all had been rendered useless.

When it is correct that mathematics can only offer us what was already in our premises, then how can it be that his conclusion—that mathematics is a quite foolish and superfluous business—is wrong?

The answer is provided in an equally simple remark: only mathematics can show us what was already in our premises!

The relation between mathematics and empirical science notably is that mathematics creates *models* and what these models are models of—well, that can be shown only by empirical means.

When I say that mathematics creates models then that is in a *wide* sense of the word. Spatial does not always correspond to spatial. It is in a highly general sense a structure which depicts a structure of things.

How can one term mathematical formulae and expressions structure; the structure is not what is put down on paper.

The structure emerges from the *rules* for the symbols on the paper. *Rules* according to which the symbols can be *rearranged*.

With mathematically described structures the rules can be made concrete by making a mechanical model which rearranges the symbols for you. However, in general the principle is that you yourself are obliged to manipulate the symbols and this *game*—given by the *symbols* and the rules—that *is* the model.

In this audience these obviously are self-evident truths. I am sketching this very general idea for you because I find that it is the correct answer to the popular mix of mathematics and science or inappreciation of what mathematics offers.

I shall now narrow my subject considerably.

There are certain kinds of *structures* which are not produced for mathematical reasons or for utilisation as technical models. These are what are popularly referred to as *games*, made for entertainment, as the basis of a formalised battle—but it remains an equally valid piece of mathematical structure.

In games—in this everyday narrower sense—it is characteristic that the structures are manipulated by 2 or more, up to several, players.

The structures of the games are mathematically highly different—what they share are completely different qualities.

By observing a number of existing games and contemplating the idea of the game—the concept of games—I have come up with the following 6 requisites which games *must*—*ought to* respectively—satisfy. And the degree to which they satisfy them at least is an indication of the game's value as game.

1. fair
2. progressive
3. final
4. *easy to comprehend*
5. strategic
6. decisive

Most games: Increasing points. Pieces forcing others back and advancing.

**I** *The idea* of utilising the quality of the plane that no more than 4 points connect.
The Four Colour Conjecture
A toric ring 7

**II** Quantify
No moves—but paper and pencil
The chronology is shown on only one board.

First player win provable unlike chess. I believed second player could win.

| Strategy |
|---|
| Offensive |
| Counteroffensive |

Tasks [Apparently to be posed during the lecture]

Not comparable with chess quite amusing that a game can be created which in principle is so simple yet is impossible to master.

Does not the poet say:

[The manuscript is clearly unfinished. Hein must have improvised the rules of Hex and the tasks or have had them elsewhere.]

### B.2.2 Demonstration

The point of this demonstration evening and this pompous scenery before us is to provide some of that which cannot be provided in newspaper articles or in a printed set of instructions, viz the direct contact with the game. Here we can talk about it refer to it. We are with the game.

I must preface this by saying that I am by no means an excellent Polygon player and I do not feel obliged to be so merely because I happen to be the inventor of the game.

I will shed theoretical light on the game and its strategy whereupon a really good player will handle the practice of the game and play it with 1 or more of the audience as time allows so that we will be able to see the characteristic strategic elements of the game in operation.

> [Crossed out]
> I would like to welcome you all to this demonstration of Polygon.
>
> I hope we shall succeed in providing you with just that which cannot be provided in brief newspaper articles or in a printed set of instructions, viz the direct contact with the game in which one can constantly refer to the game itself, really show it.

I shall begin with the beginning, not only for the sake of the beginners but to give some order to the explication. Advanced players too may benefit from hearing the motive forces systematically described and possibly in a different way than they have formulated them themselves.

The very first point is: why must the game look just like that?

When embarking on a new game I believe one asks of oneself: Is it not quite arbitrary. . . ? Could it not just as well. . . ? Why must I adhere to such an arbitrary practice?

With regards to this game I can put your mind at ease. It is not arbitrary. By its very idea this game could look no other way.

It came into being—stumbled into being—thus:

For a number of years it has been my hobby to consider the structure of the various (ex.) games and the possibilities for new games.

This has resulted in a long series of requirements which must be satisfied by any good game—and in my discovering a new kind of opportunity which had not yet been utilised. This list of general requirements is a highly disparate throng, the first one being that the players for all intents and purposes must be equal (to which we shall return).

Then follows a number of less simple demands, e.g. that a game must be progressive, it ought not to tend to move in circles; it must be final, i.e. after a limited period or number of moves this progression must reach its conclusion; it must be easy to comprehend which is a more subtle requirement meaning practically that no move is to stir up the situation, turning all advantages to disadvantages and vice versa, making it impossible to plan regularly winning tactics. The final requirement is that the game must not have a tendency to end in a draw. And it must be versatile in its possibilities for combination. And then a game like this must be implicit (math); an explicit, mathematical solution must not be possible

The idea behind the game is this: Most games are based on. . . competing number of points or forcing each other back or reduction of the opponent's markers. However, no games have yet employed the simplest quality of a surface, that which can be described thus: Taking a square, regardless of form, and drawing through it a straight or crooked connecting line between the two opposite sides and one between the two others, the two connecting lines must cross each other. —This has not been utilised before though it is such a simple quality that it seems like an obvious quality of the surface; however, it is not; mathematical surfaces can be constructed whose connecting lines do not cross each other.

It struck me that this could be employed as the principle of a game just as well as competing number of points and markers forcing back or reducing each other. One part is given the objective of combining the two sides, the other of combing the other two. How to partition this into specific moves?

Constructing the gameboard from square cells—or triangular cells for that matter—then four or more cells will touch by their corners which will stunt the game. Imagine a squared board like a chessboard and think of 4 squares that connect each with the point of a corner in one point.

This will happen in any gameboard on which more than 3 cells connect in one point.

Therefore one has to choose a construction in which only 3 cells are connected. The simplest regular solution is the hexagonal grid. On a board like this it can be proven that the game cannot end in a stalemate. Black and White cannot both block each other. One of them must make a connection. Indeed the only obstruction to one player's connection is the other's.

The subsequent task was simply to select a suitable number of cells. With some knowledge of the game it is practically a given that it must be $11 \times 11$.

And there is no reason to delimit the rule from the general: that markers can be placed anywhere. —In order to not having to introduce 1. player and 2.

When the two halves of the idea—i.e. the crossed connections and the hexagonal grid—had found each other, not only was the idea conceived but the entire game was executed.

To an unusual degree the execution is a direct consequence of the idea. It has not been necessary for me to intrude into it. One might ask if the Jack of Diamonds or the Queen of Spades in Bezique should not be worth 50 or 100 points, if the rooks in chess ought not to only move 4 squares at a time, or if the tricks in Bridge should be decided by the suit of the cards—but when the idea of Polygon is given then the entire look and law of the game has been settled. There is nothing arbitrary. I think it must be a nice thought for players of a new game that the inventor's capricious whims did not dictate the laws to which one submits but that the game crystallised from its idea in the only possible form.

Unlike a multitude of ideas for games that had only theoretical interest and ended up in my desk drawer, it turned out that this game to a very high degree satisfied the previously mentioned requirements that must be demanded of a game.

### B.2.3 Play with different people

Initially one tends to have the impression that the player with the first move—White—has a great advantage. Indeed, many people have even jumped to the conclusion that White is always able to win in very few moves.

The misconception rests in the players at this stage not yet having properly learned to cross each others paths: to fully exploit counteroffensives.

Upon learning to execute a powerful counteroffensive against the first move, one might for a time believe that it is advantageous *not* to begin because Black immediately can perform an obstructing countermove.

However, White has the same advantage over Black in his next move! Etc.—

Precisely this continuous alternation of counteroffensives is what ensures that none of the players can travel a direct route but rather that the paths entangle, possibilities branch off and the game comes alive.

But if you believe you have found a strategy which allows White to always win, you ought to calm yourself with the knowledge that gaming experts have tested the game for months and have reached the conclusion that in practice there is no discernible difference between the situations of White and Black, and that the absolute command of the game in all probability lies beyond the *practically* possible.

A teasing little quality in Polygon is that with regards to this game, unlike most other games, it can be proved that whoever begins can always win *in theory*, that is if he were able to foresee all the possibilities of the game.

The only manner of testing a "surefire winning system" is to play with other, unfamiliar players. People who have learned Polygon independently from you have often developed a completely different strategy, frequently including counteroffensives which your strategy has not considered.

It is impossible to even begin to appreciate the manifold possibilities of the game by only playing it with the same opponent or with oneself.

The road to mastering Polygon is to play with as many as different people as possible.

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# Index